

**ON THE DENSITIES OF THE LIMITING
DISTRIBUTIONS FOR QUICKSORT AND QUICKQUANT**

by

Wei-Chun Hung

A dissertation submitted to The Johns Hopkins University in conformity with the
requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

April, 2021

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Abstract

In this dissertation, we study in depth the limiting distribution of the costs of running the randomized sorting algorithm **QuickSort** and the randomized selection algorithm **QuickQuant** when the cost of sorting/selecting is measured by the number of key comparisons. It is well established in the literature that the limiting distribution F of the centered and scaled number of key comparisons required by **QuickSort** is infinitely differentiable and that the corresponding density function f enjoys superpolynomial decay in both tails. The first contribution of this dissertation is to establish upper and lower asymptotic bounds for the left and right tails of f that are nearly matching in each tail.

The literature study of the scale-normalized number of key comparisons used by the algorithm **QuickQuant**(t) for $0 \leq t \leq 1$, on the other hand, is somewhat limited and focuses on (non-limiting and limiting) moments and the limiting distribution function F_t . In particular, except knowing that $t = 0$ and $t = 1$ corresponds to the well-known Dickman distribution, from the literature we do not know much about smoothness or decay properties of F_t for $0 < t < 1$ except that $1 - F_t$ enjoys superexponential decay in the right tail. For $t \in (0, 1)$, the second contribution of this dissertation is to prove that F_t has a Lipschitz continuous density function f_t that is bounded above (by 10). We establish several fundamental properties of f_t including

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positivity of $f_t(x)$ for every $x > \min\{t, 1 - t\}$ and infinite right differentiability at $x = t$. In particular, we prove that the survival function $1 - F_t(x)$ and the density function $f_t(x)$ both have the right-tail asymptotics $\exp[-x \ln x - x \ln \ln x + O(x)]$.

The third contribution of this dissertation is to study large deviations of the number of key comparisons needed for both algorithms by using knowledge of the limiting distribution. In particular, we sharpen the large-deviation results of **QuickSort** established by McDiarmid and Hayward (1996) and produce similar new (as far as we know) results for **QuickQuant**.

Primary Reader and Advisor: James A. Fill

Secondary Reader: John C. Wierman

Acknowledgments

First, I am deeply indebted to my advisor Dr. Jim Fill for his support, encouragement, and guidance in the past several years; he has inspired me greatly not only in my academic studies but also in some perspectives on life. I also would like to express my deepest appreciation to the other members of my defense committee: Drs. John Wierman, Fred Torcaso, and Avanti Athreya. I thank them for their valuable time and their input on my Ph.D. work.

Second, I would also like to extend my gratitude to my friends at Hopkins. I thank especially my colleagues Dr. Percy Li, Dr. Shangsi Wang, Dr. Joshua Cape, Chu-Chi Li, Hsi-Wei Hsieh, Ke Chen, and Ao Sun, who provided valuable and insightful opinions for scientific discussions; and Yi-Shan Shir, Yen-Lin Chiu, Chin-Fu Liu, Dwight Meng, Tingting Bao, and Xiaokun Yu, for emotional support.

Lastly, I express my deepest gratitude to my family. I am especially grateful to Dr. Wen-Fang Chou, who has provided unparalleled support and always has faith in me. This dissertation is dedicated to all of my family members.

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Chapter 1

QuickSort: Introduction and preliminaries

1.1 Introduction

In the first part of this dissertation, we study the continuous density function f of the limiting **QuickSort** distribution. The work in Chapters 1–3 has been published as [10], [11], [12] and [13], and parts of those publications have been repeated here verbatim with the permission of both authors and the publishers. Chapters 2 through 3 reflect improvements as we discovered them. Ordinarily, an author might skip earlier improvements and proceed directly to the latest ones. We have not done so, and our choice is not motivated by any desire to pad the length of the dissertation. Rather, we believe that it is easiest for the reader to understand the more refined results with their more intricate proofs once she or he has “warmed up” with earlier results.

QuickSort is a randomized algorithm which sorts a list of n numbers (called keys), assumed here to be distinct. The algorithm first selects a key (called the pivot)

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uniformly at random and then compares every other key to the pivot, which results in $n - 1$ comparisons. The n distinct numbers can then be separated into three sublists. The first consists of those keys with values smaller than the pivot; the second consists of just the pivot key; and the third consists of those keys with values larger than the pivot. The algorithm then recursively applies the same partitioning step to the two sublists not containing the pivot until all sublists have size equal to 0 or 1, at which point the list is sorted.

Let X_n denote the (random) number of comparisons when sorting n distinct numbers using the algorithm `QuickSort`. Clearly $X_0 = 0$, and for $n \geq 1$ we have the recurrence relation

$$X_n \stackrel{\mathcal{L}}{=} X_{U_n-1} + X_{n-U_n}^* + n - 1,$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law (i.e., in distribution); $X_k \stackrel{\mathcal{L}}{=} X_k^*$; the random variable U_n is uniformly distributed on $\{1, \dots, n\}$; and $U_n, X_0, \dots, X_{n-1}, X_0^*, \dots, X_{n-1}^*$ are all independent. It is well known that

$$\mathbb{E}X_n = 2(n+1)H_n - 4n,$$

where H_n is the n th harmonic number $H_n := \sum_{k=1}^n k^{-1}$ and (from a simple exact expression) that $\text{Var } X_n = (1 + o(1))(7 - \frac{2\pi^2}{3})n^2$. To study distributional asymptotics, we first center and scale X_n as follows:

$$Z_n = \frac{X_n - \mathbb{E}X_n}{n}. \tag{1.1}$$

Using the Wasserstein d_2 -metric, Rösler [35] proved that Z_n converges to Z weakly as $n \rightarrow \infty$. Using a martingale argument, Régnier [34] proved that the slightly renormalized $\frac{n}{n+1}Z_n$ converges to Z in L^p for every finite p , and thus in distribution;

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equivalently, the same conclusions hold for Z_n . The random variable Z has everywhere finite moment generating function with $\mathbb{E}Z = 0$ and $\text{Var } Z = 7 - (2\pi^2/3)$. Moreover, Z satisfies the distributional identity

$$Z \stackrel{\mathcal{L}}{=} UZ + (1 - U)Z^* + g(U). \quad (1.2)$$

On the right, $Z^* \stackrel{\mathcal{L}}{=} Z$; U is uniformly distributed on $(0, 1)$; U, Z, Z^* are independent; and

$$g(u) := 2u \ln u + 2(1 - u) \ln(1 - u) + 1.$$

Further, the distributional identity together with the condition that $\mathbb{E}Z$ (exists and) vanishes characterizes the limiting **Quicksort** distribution; this was first shown by Rösler [35] under the additional condition that $\text{Var } Z < \infty$, and later in full by Fill and Janson [14].

Fill and Janson [15] derived basic properties of the limiting **QuickSort** distribution $\mathcal{L}(Z)$. In particular, they proved that $\mathcal{L}(Z)$ has a (unique) continuous density f which is everywhere positive and infinitely differentiable, and for every $k \geq 0$ that $f^{(k)}$ is bounded and enjoys superpolynomial decay in both tails, that is, for each $p \geq 0$ and $k \geq 0$ there exists a finite constant $C_{p,k}$ such that $|f^{(k)}(x)| \leq C_{p,k}|x|^{-p}$ for all $x \in \mathbb{R}$.

In this dissertation, we study asymptotics of $f(-x)$ and $f(x)$ as $x \rightarrow \infty$. Janson [24] concerned himself with the corresponding asymptotics for the distribution function F and wrote this: “Using non-rigorous methods from applied mathematics (assuming an as yet unverified regularity hypothesis), Knessl and Szpankowski [25] found very precise asymptotics of both the left tail and the right tail.” Janson specifies these Knessl–Szpankowski asymptotics for F in his equations (1.6)–(1.7). But Knessl and Szpankowski actually did more, producing asymptotics for f , which were

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integrated by Janson to get corresponding asymptotics for F . We utilize the same abbreviation $\gamma := (2 - \frac{1}{\ln 2})^{-1}$ as Janson [24]. With the *same* constant c_3 as in (1.6) of [24], the density analogues of (1.6) (omitting the middle expression) and (1.7) of [24] are that, as $x \rightarrow \infty$, Knessl and Szpankowski [25] find

$$f(-x) = \exp \left[-e^{\gamma x + c_3 + o(1)} \right] \quad (1.3)$$

for the left tail and

$$f(x) = \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)] \quad (1.4)$$

for the right tail.

We will come as close to these non-rigorous results for the density as Janson [24] does for the distribution function, and we also obtain similar asymptotic bounds for tail suprema of absolute values of derivatives of the density. Although our asymptotics for f imply the asymptotics for F in Janson's main Theorem 1.1, it is important to note that in the case of upper bounds (but not lower bounds) on f we use his results in the proofs of ours.

The next two theorems are our main results of this chapter.

Theorem 1.1. *Let $\gamma := (2 - \frac{1}{\ln 2})^{-1}$. As $x \rightarrow \infty$, the limiting Quicksort density function f satisfies*

$$\exp \left[-e^{\gamma x + \ln \ln x + O(1)} \right] \leq f(-x) \leq \exp \left[-e^{\gamma x + O(1)} \right], \quad (1.5)$$

$$\exp[-x \ln x - x \ln \ln x + O(x)] \leq f(x) \leq \exp[-x \ln x + O(x)]. \quad (1.6)$$

To state our second main theorem we let $\underline{F}(x) := F(-x)$ and $\overline{F}(x) := 1 - F(x)$,

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and for a function $h : \mathbb{R} \rightarrow \mathbb{R}$ we write

$$\|h\|_x := \sup_{t \geq x} |h(t)|. \quad (1.7)$$

Theorem 1.2. *Given an integer $k \geq 0$, as $x \rightarrow \infty$ the k^{th} derivative of the limiting QuickSort distribution function F satisfies*

$$\exp[-e^{\gamma x + \ln \ln x + O(1)}] \leq \|\underline{F}^{(k)}\|_x \leq \exp[-e^{\gamma x + O(1)}], \quad (1.8)$$

$$\exp[-x \ln x - (k \vee 1)x \ln \ln x + O(x)] \leq \|\overline{F}^{(k)}\|_x \leq \exp[-x \ln x + O(x)]. \quad (1.9)$$

Remark 1.3. (a) Using the monotonicity of F , it is easy to see that the assertions of Theorem 1.2 for $k = 0$ are equivalent to the main Theorem 1.1 of Janson [24], which agrees with the formulation of our Theorem 1.2 in that case except that the four bounds are on $|\underline{F}(x)|$ and $|\overline{F}(x)|$ instead of the tail suprema $\|\underline{F}\|_x$ and $\|\overline{F}\|_x$. Further, our Theorem 1.1 implies the assertions of Theorem 1.2 for $k = 1$. So we need only prove Theorem 1.1 and Theorem 1.2 for $k \geq 2$.

(b) The non-rigorous arguments of Knessl and Szpankowski [25] suggest that the following asymptotics as $x \rightarrow \infty$ obtained by repeated formal differentiation of (1.3)–(1.4) are correct for every $k \geq 0$:

$$f^{(k)}(-x) = \exp[-e^{\gamma x + c_3 + o(1)}], \quad (1.10)$$

$$f^{(k)}(x) = (-1)^k \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)]. \quad (1.11)$$

But these remain conjectures for now. Unfortunately, for $k \geq 1$ we don't even know how to identify rigorously the asymptotic signs of $f^{(k)}(\mp x)$! Concerning $k = 1$, it has long been conjectured that f is unimodal. This would of course imply that $f'(-x) > 0$

and $f'(x) < 0$ for sufficiently large x .

As already mentioned, Fill and Janson [15] proved that for each $p \geq 0$ and $k \geq 0$ there exists a finite constant $C_{p,k}$ such that $|f^{(k)}(x)| \leq C_{p,k}|x|^{-p}$ for all $x \in \mathbb{R}$. Our technique for proving the upper bounds in Theorems 1.1 and 1.2 is to use explicit bounds on the constants $C_k := C_{0,k}$ together with the Landau–Kolmogorov inequality (see, for example, [36]).

1.2 Preliminaries

1.2.1 An integral equation for f

Fill and Janson [15, Theorem 4.1 and (4.2)] produced an integral equation satisfied by f , namely,

$$f(x) = \int_{u=0}^1 \int_{z \in \mathbb{R}} f(z) f\left(\frac{x - g(u) - (1-u)z}{u}\right) \frac{1}{u} dz du. \quad (1.12)$$

This integral equation will be used in the proofs of our lower-bound results for f .

1.2.2 Landau–Kolmogorov inequality

For an overview of the Landau–Kolmogorov inequality, see [33, Chapter 1]. Here we state a version of the inequality well-suited to our purposes; see [30] and [36, display (21) and the display following (17)].

Lemma 1.4. *Let $n \geq 2$, and suppose $h : \mathbb{R} \rightarrow \mathbb{R}$ has n derivatives. If h and $h^{(n)}$ are both bounded, then for $1 \leq k < n$ so is $h^{(k)}$. Moreover, there exist constants $c_{n,k}$ (not depending on h) such that, for every $x \in \mathbb{R}$, the supremum norm $\|\cdot\|_x$ defined*

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at (1.7) satisfies

$$\|h^{(k)}\|_x \leq c_{n,k} \|h\|_x^{1-(k/n)} \|h^{(n)}\|_x^{k/n}, \quad 1 \leq k < n.$$

Further, for $1 \leq k \leq n/2$ the best constants $c_{n,k}$ satisfy

$$c_{n,k} \leq n^{(1/2)[1-(k/n)]} (n-k)^{-1/2} \left(\frac{e^2 n}{4k} \right)^k \leq \left(\frac{e^2 n}{4k} \right)^k.$$

1.2.3 Explicit constant upper bounds for absolute derivatives

We also make use of the following two results extracted from [15, Theorem 2.1 and (3.3)].

Lemma 1.5. *Let ϕ denote the characteristic function corresponding to f . Then for every real $p \geq 0$ we have*

$$|\phi(t)| \leq 2^{p^2+6p} |t|^{-p} \quad \text{for all } t \in \mathbb{R}.$$

Lemma 1.6. *For every integer $k \geq 0$ we have*

$$\sup_{x \in \mathbb{R}} |f^{(k)}(x)| \leq \frac{1}{2\pi} \int_{t=-\infty}^{\infty} |t|^k |\phi(t)| dt.$$

Using these two results, it is now easy to bound $f^{(k)}$.

Proposition 1.7. *For every integer $k \geq 0$ we have*

$$\sup_{x \in \mathbb{R}} |f^{(k)}(x)| \leq 2^{k^2+10k+17}.$$

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Proof. For every integer $k \geq 0$ we have

$$\begin{aligned}
 \sup_{x \in \mathbb{R}} |f^{(k)}(x)| &\leq \frac{1}{2\pi} \int_{t=-\infty}^{\infty} |t|^k |\phi(t)| \, dt \\
 &\leq \frac{1}{2\pi} \left[\int_{|t|>1} |t|^k |\phi(t)| \, dt + \int_{|t|\leq 1} |t|^k |\phi(t)| \, dt \right] \\
 &\leq \frac{1}{2\pi} \left[\int_{|t|>1} 2^{(k+2)^2+6(k+2)} t^{-2} \, dt + \int_{|t|\leq 1} |t|^k \, dt \right] \\
 &\leq \frac{1}{\pi} \left[2^{k^2+10k+16} + \frac{1}{k+1} \right] \leq 2^{k^2+10k+17},
 \end{aligned}$$

as desired. □

Chapter 2

On the tails of the limiting QuickSort density

2.1 Left tail lower bound on f

Our iterative approach to finding the left tail lower bound on f in Theorem 1.1 is similar to the method used by Janson [24] for F . The following lemma gives us an inequality that is essential in this section; as we shall see, it is established from a recurrence inequality. For $z \geq 0$ define

$$m_z := \left(\min_{x \in [-z, 0]} f(x) \right) \wedge 1.$$

Lemma 2.1. *Given $\epsilon \in (0, 1/10)$, let $a \equiv a(\epsilon) := -g\left(\frac{1}{2} - \epsilon\right) > 0$. Then for any integer $k \geq 2$ we have*

$$m_{ka} \geq (2\epsilon^3 m_{2a})^{2^{k-2}}.$$

We delay the proof of Lemma 2.1 in order to show next how the lemma leads us to the desired lower bound in (1.5) on the left tail of f by using the same technique

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as in [24] for F .

Proposition 2.2. *As $x \rightarrow \infty$ we have*

$$\ln f(-x) \geq -e^{\gamma x + \ln \ln x + O(1)}.$$

Proof. By Lemma 2.1, for $x > a$ we have

$$f(-x) \geq m_x \geq m\left(\left\lceil \frac{x}{a} \right\rceil a\right) \geq (2\epsilon^3 m_{2a})^{2^{\lceil x/a \rceil - 2}} \geq (2\epsilon^3 m_{2a})^{2^{x/a}},$$

provided ϵ is sufficiently small that $2\epsilon^3 m_{2a} < 1$. The same as Janson [24], we pick $\epsilon = x^{-1/2}$ and, setting $\gamma = (2 - \frac{1}{\ln 2})^{-1}$, get $\frac{1}{a} = \frac{\gamma}{\ln 2} + O(x^{-1})$ and

$$\begin{aligned} \ln f(-x) &\geq 2^{\frac{\gamma}{\ln 2}x + O(1)} \cdot \ln(2\epsilon^3 m_{2a}) \\ &= e^{\gamma x + O(1)} \cdot \left(-\frac{3}{2} \ln x + \ln m_{2a} + \ln 2\right) \\ &\geq -e^{\gamma x + \ln \ln x + O(1)}. \end{aligned}$$

□

Now we go back to prove Lemma 2.1:

Proof of Lemma 2.1. By the integral equation (1.12) satisfied by f (and symmetry in u about $u = 1/2$), for arbitrary z and a we have

$$f(-z - a) = 2 \int_{u=0}^{1/2} \int_{y \in \mathbb{R}} f(y) f\left(\frac{-z - a - g(u) - (1-u)y}{u}\right) \frac{1}{u} dy du. \quad (2.1)$$

Since f is everywhere positive, we can get a lower bound on $f(-z - a)$ by restricting

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the range of integration in (2.1). Therefore,

$$f(-z-a) \geq 2 \int_{u=\frac{1}{2}-\frac{\epsilon}{2}}^{1/2} \int_{y=-z}^{-z+\epsilon^2} f(y) f\left(\frac{-z-a-g(u)-(1-u)y}{u}\right) \frac{1}{u} dy du. \quad (2.2)$$

We claim that in this integral region, we have $\frac{-z-a-g(u)-(1-u)y}{u} \geq -z$, which is equivalent to $y+z \leq \frac{-a-g(u)}{1-u}$. Here is a proof. Observe that when ϵ is small enough and $u \in [\frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2}]$, we have

$$\begin{aligned} \frac{-a-g(u)}{1-u} &\geq \frac{g\left(\frac{1}{2}-\epsilon\right)-g\left(\frac{1}{2}-\frac{\epsilon}{2}\right)}{\frac{1}{2}+\frac{\epsilon}{2}} \\ &\geq \frac{\frac{\epsilon}{2} |g'\left(\frac{1}{2}-\frac{\epsilon}{2}\right)|}{\frac{1}{2}+\frac{\epsilon}{2}} = \frac{\epsilon}{1+\epsilon} \left| 2 \ln \left(1 - \frac{2\epsilon}{1+\epsilon} \right) \right| \\ &\geq \frac{4\epsilon^2}{(1+\epsilon)^2} \geq \epsilon^2. \end{aligned}$$

Also, in this integral region we have $y+z \leq \epsilon^2$. So we conclude that $y+z \leq \frac{-a-g(u)}{1-u}$.

Next, we claim that $\frac{-z-a-g(u)-(1-u)y}{u} \leq 0$ in this integral region if z is large enough. Here is a proof. Let $\frac{-z-a-g(u)-(1-u)y}{u} = -z+\delta$ with $\delta \geq 0$. Then in the integral region we have $0 \leq y+z = \frac{-a-g(u)-u\delta}{1-u}$. Therefore

$$\begin{aligned} \delta &\leq \frac{-a-g(u)}{u} \leq \frac{-a-g\left(\frac{1}{2}\right)}{\frac{1}{2}-\frac{\epsilon}{2}} = \frac{2}{1-\epsilon} \left[g\left(\frac{1}{2}-\epsilon\right) - g\left(\frac{1}{2}\right) \right] \\ &\leq \frac{2\epsilon}{1-\epsilon} \left| 2 \ln \left(1 - \frac{4\epsilon}{1+2\epsilon} \right) \right| \\ &\leq 19\epsilon^2, \end{aligned}$$

where the last inequality can be verified to hold for $\epsilon < 1/10$. That means if we pick z large enough, for example, $z \geq 20\epsilon^2$, then $\frac{-z-a-g(u)-(1-u)y}{u} = -z+\delta$ will be negative.

It can also be verified that $a \geq 30\epsilon^2$ for $\epsilon < 1/10$.

Now consider $\epsilon < 1/10$, an integer $k \geq 3$, $z \in [(k-2)a, (k-1)a]$, and $x = z+a \in$

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$[(k-1)a, ka]$. Noting $z \geq a \geq 30\epsilon^2 > 20\epsilon^2$, by (2.2) we have

$$f(-x) \geq 2 \cdot \frac{\epsilon}{2} \cdot m_z^2 \cdot \epsilon^2 \cdot 2 \geq 2\epsilon^3 m_{(k-1)a}^2.$$

Further, for $x \in [0, (k-1)a]$ we have

$$f(-x) \geq m_{(k-1)a} > 2\epsilon^3 m_{(k-1)a}^2$$

since $2\epsilon^3 < 1$ and $m_{(k-1)a} \leq 1$ by definition. Combine these two facts, we can conclude that for $x \in [0, ka]$ we have $f(-x) \geq 2\epsilon^3 m_{(k-1)a}^2$. This implies the recurrence inequality

$$m_{ka} \geq 2\epsilon^3 m_{(k-1)a}^2.$$

The desired inequality follows by iterating:

$$m_{ka} \geq (2\epsilon^3)^{2^{k-2}-1} m_{2a}^{2^{k-2}} \geq (2\epsilon^3 \cdot m_{2a})^{2^{k-2}}.$$

□

2.2 Right tail lower bound on f

Once again we use an iterative approach to derive our right-tail lower bound on f in Theorem 1.1. The following key lemma is established from a recurrence inequality. Define

$$c := 2[F(1) - F(0)] \in (0, 2)$$

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and

$$m_z := \min_{x \in [0, z]} f(x), \quad z \geq 0.$$

Lemma 2.3. *Suppose $b \in [0, 1)$ and that $\delta \in (0, 1/2)$ is sufficiently small that $g(\delta) \geq b$. Then for any integer $k \geq 1$ satisfying*

$$2 + (k - 1)b \leq [g(\delta) - b]/\delta$$

we have

$$m_{2+kb} \geq (c\delta)^{k-1} m_3.$$

We delay the proof of Lemma 2.3 in order to show next how the lemma leads us to the desired lower bound in (1.6) on the right tail of f .

Proposition 2.4. *As $x \rightarrow \infty$ we have*

$$f(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. Given $x \geq 3$ suitably large, we will show next that we can apply Lemma 2.3 for suitably chosen $b > 0$ and δ and $k = \lceil (x - 2)/b \rceil \geq 2$. Then, by the lemma,

$$f(x) \geq m_{2+kb} \geq (c\delta)^{k-1} m_3 \geq (c\delta)^{(x-2)/b} m_3, \tag{2.3}$$

and we will use (2.3) to establish the proposition.

We make the same choices of δ and b as in [24, Sec. 4], namely, $\delta = 1/(x \ln x)$ and $b = 1 - (2/\ln x)$. To apply Lemma 2.3, we need to check that $g(\delta) \geq b$ and $2 + (k - 1)b \leq [g(\delta) - b]/\delta$, for the latter of which it is sufficient that $x \leq [g(\delta) - b]/\delta$.

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Indeed, if x is sufficiently large, then

$$g(\delta) \geq 1 + 3\delta \ln \delta = 1 - \frac{3}{x \ln x} (\ln x + \ln \ln x) \geq 1 - \frac{4}{x},$$

where the elementary first inequality is (4.1) in [24], and so

$$g(\delta) - b \geq \frac{2}{\ln x} - \frac{4}{x} \geq \frac{1}{\ln x} > 0$$

and

$$\frac{g(\delta) - b}{\delta} \geq \frac{1/\ln x}{1/(x \ln x)} = x.$$

Finally, we use (2.3) to establish the proposition. Indeed,

$$\begin{aligned} -\ln f(x) &\leq \frac{x-2}{b} \ln\left(\frac{1}{c\delta}\right) - \ln m_3 \\ &\leq \frac{x}{1-(2/\ln x)} [\ln(x \ln x) + \ln(\tfrac{1}{c})] - \ln m_3 \\ &= \frac{x}{1-(2/\ln x)} \ln(x \ln x) + O(x). \end{aligned}$$

But

$$\begin{aligned} &\frac{x}{1-(2/\ln x)} \ln(x \ln x) \\ &= x \left[1 + \frac{2}{\ln x} + O\left(\frac{1}{(\log x)^2}\right) \right] (\ln x + \ln \ln x) \\ &= (x \ln x) \left[1 + \frac{2}{\ln x} + O\left(\frac{1}{(\log x)^2}\right) \right] \left(1 + \frac{\ln \ln x}{\ln x} \right) \\ &= (x \ln x) \left[1 + \frac{\ln \ln x}{\ln x} + \frac{2}{\ln x} + \frac{2 \ln \ln x}{(\ln x)^2} + O\left(\frac{1}{(\log x)^2}\right) \right] \\ &= x \ln x + x \ln \ln x + 2x + \frac{2x \ln \ln x}{\ln x} + O\left(\frac{x}{\log x}\right) \\ &= x \ln x + x \ln \ln x + O(x). \end{aligned}$$

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So

$$-\ln f(x) \leq x \ln x + x \ln \ln x + O(x),$$

as claimed. □

Now we go back to prove Lemma 2.3, but first we need two preparatory results.

Lemma 2.5. *Suppose $z \geq 2$, $b \geq 0$, and $\delta \in (0, 1/2)$ satisfy $g(\delta) \geq b$ and $z \leq [g(\delta) - b]/\delta$. Then*

$$f(z + b) \geq c \delta m_z.$$

Proof. By the integral equation (1.12) satisfied by f (and symmetry in u about $u = 1/2$), for arbitrary z and b we have

$$f(z + b) = 2 \int_{u=0}^{1/2} \int_{y \in \mathbb{R}} f(y) f\left(\frac{z + b - g(u) - (1 - u)y}{u}\right) \frac{1}{u} dy du.$$

Since f is positive everywhere, a lower bound on $f(z + b)$ can be achieved by shrinking the region of integration:

$$\begin{aligned} f(z + b) &\geq 2 \int_{u=0}^{\delta} \int_{y=0}^z f(y) f\left(\frac{z + b - g(u) - (1 - u)y}{u}\right) \frac{1}{u} dy du \\ &\geq 2m_z \int_{u=0}^{\delta} \int_{y=0}^z f\left(\frac{z + b - g(u) - (1 - u)y}{u}\right) \frac{1}{u} dy du \\ &= 2m_z \int_{u=0}^{\delta} \int_{\xi=z+\frac{b-g(u)}{u}}^{\frac{z+b-g(u)}{u}} f(\xi) \frac{1}{1-u} d\xi du. \end{aligned} \tag{2.4}$$

The equality comes from a change of variables. We next claim that the integral of integration for ξ contains $(0, z - 1)$, and then the desired result follows. Indeed, if $u \in (0, \delta)$ and $\xi \in (0, z - 1)$ then

$$\xi < z - 1 < \frac{z-1}{u} \leq \frac{z+b-g(u)}{u},$$

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where the last inequality holds because $b \geq 0$ and $g(u) \leq 1$; and, because $g(u) \geq g(\delta)$ and $g(\delta) \geq b$ and $z \leq [g(\delta) - b]/\delta$, we have

$$\begin{aligned} \xi > 0 &= z + \frac{b-g(u)}{u} - [z + \frac{b-g(u)}{u}] \geq z + \frac{b-g(u)}{u} - [z + \frac{b-g(\delta)}{u}] \\ &\geq z + \frac{b-g(u)}{u} - [z + \frac{b-g(\delta)}{\delta}] \geq z + \frac{b-g(u)}{u}. \end{aligned}$$

□

Lemma 2.6. *Suppose $b \geq 0$ and that $\delta \in (0, 1/2)$ is sufficiently small that $g(\delta) \geq b$. Then for any integer $k \geq 2$ satisfying*

$$2 + (k-1)b \leq [g(\delta) - b]/\delta$$

we have

$$m_{2+kb} \geq c \delta m_{2+(k-1)b}.$$

Proof. For $y \in [2 + (k-1)b, 2 + kb]$, application of Lemma 2.5 with $z = y - b$ yields

$$f(y) \geq c \delta m_{y-b} \geq c \delta m_{2+(k-1)b}.$$

Also, for $y \in [0, 2 + (k-1)b]$ we certainly have

$$f(y) \geq m_{2+(k-1)b} > c \delta m_{2+(k-1)b}.$$

The result follows. □

We are now ready to complete this section by proving Lemma 2.3.

Proof of Lemma 2.3. By iterating the recurrence inequality of Lemma 2.6, it follows

that

$$m_{2+kb} \geq (c\delta)^{k-1} m_{2+b}.$$

Lemma 2.3 then follows since $b < 1$. □

2.3 Left tail bounds for tail suprema of absolute derivatives

From Section 2.1 (respectively, Section 2.2) we know the left-tail lower bound of (1.5) [resp., the right-tail lower bound of (1.6)]. In this section we establish the left-tail bounds of (1.5) and (1.8), and in the next section we do the same for right tails.

2.3.1 Lower bounds

As discussed in Remark 1.3(a), in light of the main theorem of Janson [24] and our Section 2.1, to finish our treatment of left-tail lower bounds we need only prove the lower bound in (1.8) for fixed $k \geq 2$. For that, choose any x and apply the Landau–Kolmogorov Lemma 1.4, bounding the function $\underline{F}'(\cdot) = -f(-\cdot)$ in terms of the functions \underline{F} and $\underline{F}^{(k)}$. This gives

$$f(-x) \leq \|\underline{F}'\|_x \leq c_{k,1} \|\underline{F}\|_x^{(k-1)/k} \|\underline{F}^{(k)}\|_x^{1/k},$$

i.e.,

$$\|\underline{F}^{(k)}\|_x \geq c_{k,1}^{-k} \|\underline{F}\|_x^{-(k-1)} [f(-x)]^k.$$

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But recall

$$c_{k,1} \leq e^2 k/4, \quad \|\underline{F}\|_x \leq \exp \left[-e^{\gamma x + O(1)} \right], \quad f(-x) \geq \exp \left[-e^{\gamma x + \ln \ln x + O(1)} \right].$$

Plugging in these bounds, we obtain the desired result.

2.3.2 Upper bounds

The left-tail upper bounds in (1.8) of Theorem 1.2 can be written in the equivalent form

$$\lambda_k := \limsup_{x \rightarrow \infty} \left[\gamma x - \ln \left(-\ln \|\underline{F}^{(k)}\|_x \right) \right] < \infty; \quad (2.5)$$

note also that the left-tail upper bound in (1.5) of Theorem 1.1 follows from $\lambda_1 < \infty$. As discussed in Remark 1.3(a), (2.5) is known for $k = 0$ from Janson [24]. So to finish our treatment of left-tail upper bounds in Theorems 1.1–1.2 we need only prove (2.5) for $k \geq 1$.

In this subsection we prove the following stronger Proposition 2.7, which implies that λ_k is non-increasing in $k \geq 0$ and therefore that $\lambda_k < \infty$ for every k . In preparation for the proof, see the definition of μ_j in (2.6) and note that if $\mu_j \leq 0$ for $j = 0, \dots, k-1$, then λ_j is non-increasing for $j = 0, \dots, k$; in particular, (2.5) then holds.

Proposition 2.7. *For each fixed $k \geq 0$ we have*

$$\mu_k := \limsup_{x \rightarrow \infty} \left[-\ln \left(-\ln \|\underline{F}^{(k+1)}\|_x \right) + \ln \left(-\ln \|\underline{F}^{(k)}\|_x \right) \right] \leq 0. \quad (2.6)$$

Proof. We proceed by induction on k . Choosing any x and applying the Landau–

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Kolmogorov inequality Lemma 1.4 to the function $h = \underline{F}^{(k)}$, we find for $n \geq 2$ that

$$\|\underline{F}^{(k+1)}\|_x \leq \frac{1}{4}e^2 n \|\underline{F}^{(k)}\|_x^{1-(1/n)} \|\underline{F}^{(k+n)}\|_x^{1/n}.$$

We can bound the norm $\|\underline{F}^{(k+n)}\|_x$ using Proposition 1.7 simply by

$$a_{n,k} := 2^{(k+n-1)^2+10(k+n-1)+17}. \quad (2.7)$$

Thus the argument of the lim sup in (2.6) can be bounded above by

$$-\ln \left[1 - \frac{1}{n} - \frac{2 - \ln 4 + \ln n + n^{-1} \ln a_{n,k}}{-\ln \|\underline{F}^{(k)}\|_x} \right].$$

By Janson's bound giving $\lambda_0 < \infty$ if $k = 0$ and by induction on k if $k \geq 1$, we know that (2.5) holds. Thus, letting $n \equiv n(x) \rightarrow \infty$ with $n(x) = o(e^{\gamma x})$, the claimed inequality follows. \square

Remark 2.8. According to Remark 1.3, it is natural to conjecture that for every k the lim sup in (2.5) is a limit and equals $-c_3$ and hence the lim sup in (2.6) is a vanishing limit.

2.4 Right tail bounds for tail suprema of absolute derivatives

In this section we establish the right-tail bounds of (1.6) and (1.9).

2.4.1 Lower bounds

As discussed in Remark 1.3(a), in light of the main theorem of [24] and our Section 2.2, to finish our treatment of right-tail lower bounds we need only prove the lower bound in (1.9) for fixed $k \geq 2$. For that, proceed using the Landau–Kolmogorov Lemma 1.4 as in Section 2.3.1 to obtain

$$\|\overline{F}^{(k)}\|_x \geq c_{k,1}^{-k} \|\overline{F}\|_x^{-(k-1)} [f(x)]^k.$$

But recall

$$\begin{aligned} c_{k,1} &\leq e^2 k/4, \quad \|\overline{F}\|_x \leq \exp[-x \ln x + O(x)], \\ f(x) &\geq \exp[-x \ln x - x \ln \ln x + O(x)]. \end{aligned}$$

Plugging in these bounds, we obtain the desired result.

2.4.2 Upper bounds

The right-tail upper bounds in (1.9) of Theorem 1.2 can be written in the equivalent form

$$\rho_k := \limsup_{x \rightarrow \infty} x^{-1} \left(x \ln x + \ln \|\overline{F}^{(k)}\|_x \right) < \infty; \quad (2.8)$$

note also that the right-tail upper bound in (1.6) of Theorem 1.1 follows from $\rho_1 < \infty$. As discussed in Remark 1.3(a), (2.8) is known for $k = 0$ from Janson [24]. So to finish our treatment of right-tail upper bounds in Theorems 1.1–1.2 we need only prove (2.8) for $k \geq 1$.

In this subsection we prove the next stronger Proposition 2.9, a right-tail analogue of Proposition 2.7, and it then follows by choosing $r(x) \equiv x$ that ρ_k is non-increasing

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in $k \geq 0$ and therefore that $\rho_k < \infty$ for every k .

Proposition 2.9. *Let r be a function satisfying $r(x) = \omega(\sqrt{x \log x})$ as $x \rightarrow \infty$. Then for each fixed $k \geq 0$ we have*

$$\sigma_k := \limsup_{x \rightarrow \infty} r(x)^{-1} \left(\ln \|\bar{F}^{(k+1)}\|_x - \ln \|\bar{F}^{(k)}\|_x \right) \leq 0. \quad (2.9)$$

Proof. Proceeding as in the proof of Proposition 2.7, for any x and any $n \geq 2$ we have

$$\|\bar{F}^{(k+1)}\|_x \leq \frac{1}{4} e^2 n \|\bar{F}^{(k)}\|_x^{1-(1/n)} \|\bar{F}^{(k+n)}\|_x^{1/n},$$

we again bound the norm $\|\bar{F}^{(k+n)}\|_x$ by (2.7). Thus the argument of the lim sup in (2.9) can be bounded above by

$$r(x)^{-1} \left[\frac{1}{n} (-\ln \|\bar{F}^{(k)}\|_x) + 2 - \ln 4 + \ln n + \frac{1}{n} \ln a_{n,k} \right].$$

By the right-tail *lower* bound for $\|\bar{F}^{(k)}\|_x$ in (1.9) (established in the preceding subsection), we know that

$$-\ln \|\bar{F}^{(k)}\|_x \leq x \ln x + (k \vee 1)x \ln \ln x + O(x) = (1 + o(1))x \ln x.$$

Thus, letting $n \equiv n(x)$ satisfy $n(x) = \omega((x \log x)/r(x))$ and $n(x) = o(r(x))$, the claimed inequality follows. \square

Remark 2.10. According to Remark 1.3, it is natural to conjecture that for every k we have $\rho_k = -\infty$ and the lim sup in (2.9) with $r(x) \equiv x$ is a vanishing limit.

Chapter 3

QuickSort: Improved right-tail asymptotics for the limiting distribution, and large deviations

3.1 Improved right-tail asymptotics for the limiting distribution

We now focus on improving the right-tail upper bound for the survival function $1 - F(x)$ and its derivatives. As discussed in [24, Section 1] and in Remark 1.3, non-rigorous arguments of Knessl and Szpankowski [25] suggest very refined asymptotics, which to three logarithmic terms assert that for each $k \geq 0$ we have

$$\overline{F}^{(k)}(x) = \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)] \quad (3.1)$$

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as $x \rightarrow \infty$ (and hence that the same asymptotics hold for $\|\overline{F}^{(k)}\|_x$). Note that for $k = 0, 1$ these expansions match the lower bounds on f and \overline{F} in (1.6) and (1.9) to two logarithmic terms.

In this chapter, we refine the upper bounds of (1.6) and (1.9) to match (3.1), and we are also able to improve the lower bound in (1.9) to match (3.1) to two terms. Here is our main theorem:

Theorem 3.1. (a) *As $x \rightarrow \infty$, the limiting QuickSort density function f satisfies*

$$\exp[-x \ln x - x \ln \ln x + O(x)] \leq f(x) \quad (3.2)$$

$$\leq \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)]. \quad (3.3)$$

(b) *Given an integer $k \geq 0$, as $x \rightarrow \infty$ the k^{th} derivative of the limiting QuickSort distribution function F satisfies*

$$\exp[-x \ln x - x \ln \ln x + O(x)] \leq \|\overline{F}^{(k)}\|_x \quad (3.4)$$

$$\leq \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)]. \quad (3.5)$$

We next argue that to prove our main Theorem 3.1 we need only establish the following equivalent version of the upper bound (3.5) in the case $k = 0$:

Proposition 3.2. *As $x \rightarrow \infty$, the limiting QuickSort distribution function F satisfies*

$$\overline{F}(x) \leq \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)].$$

Here is the argument proving Theorem 3.1 from Proposition 3.2. We already know (3.2) and, for $k = 0, 1$, the lower bounds (3.4) from (1.6) and (1.9). Next, the upper bounds in (3.5) for general values of k follow inductively from Proposition 3.2

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using Proposition 2.9 with $r(x) \equiv x$, according to which

$$\limsup_{x \rightarrow \infty} x^{-1} \left(\ln \|\overline{F}^{(k+1)}\|_x - \ln \|\overline{F}^{(k)}\|_x \right) \leq 0.$$

The upper bound (3.3) follows immediately from (3.5) with $k = 1$. Finally, the lower bounds (3.4) for fixed $k \geq 2$ follow as in Section 2.4.1, using the improved upper bound of Proposition 3.2. To spell this out, as in Section 2.4.1, the Landau–Kolmogorov inequality (see Lemma 1.4) implies

$$\|\overline{F}^{(k)}\|_x \geq c_{k,1}^{-k} \|\overline{F}\|_x^{-(k-1)} [f(x)]^k,$$

where it is already known from Lemma 1.4 and (1.6) that

$$c_{k,1} \leq e^2 k / 4$$

and

$$f(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)].$$

Moreover, now we have from Proposition 3.2 the improved upper bound (3.5) on $\|\overline{F}\|_x$. Plugging in these bounds, we obtain (3.4).

Remark 3.3. It follows immediately from our improved upper bound (3.5) that the first conjecture in Remark 2.10 that

$$\rho_k := \lim_{x \rightarrow \infty} x^{-1} \left(x \ln x + \ln \|\overline{F}^{(k)}\|_x \right) = -\infty$$

is true. However, because the third term in (3.5) is not matched in (3.4), the second

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conjecture in that remark, namely, that

$$\lim_{x \rightarrow \infty} x^{-1} \left(\ln \|\bar{F}^{(k+1)}\|_x - \ln \|\bar{F}^{(k)}\|_x \right) = 0,$$

remains unproved.

We conclude this section with an open problem concerning *left*-tail behavior.

Open Problem. With $\underline{F}(x) := F(-x)$ can the *lower* bounds as $x \rightarrow \infty$ in the left-tail results

$$\exp \left[-e^{\gamma x + \ln \ln x + O(1)} \right] \leq f(-x) \leq \exp \left[-e^{\gamma x + O(1)} \right], \quad (3.6)$$

$$\exp \left[-e^{\gamma x + \ln \ln x + O(1)} \right] \leq \|\underline{F}^{(k)}\|_x \leq \exp \left[-e^{\gamma x + O(1)} \right] \quad (3.7)$$

of [24] and (1.5) be improved to match the asymptotics

$$\underline{F}^{(k)}(x) = \exp \left[-e^{\gamma x + O(1)} \right]$$

suggested by Knessl and Szpankowski [25], where $\gamma := (2 - \frac{1}{\ln 2})^{-1}$?

3.2 Proof of the main Proposition 3.2

Let ψ denote the moment generating function of Z . It was shown by Rösler [35] that ψ is everywhere finite. As we next show, Proposition 3.2 follows easily by (i) combining the Chernoff bound

$$\bar{F}(x) = \mathbb{P}(Z \geq x) \leq e^{-tx} \psi(t),$$

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choosing $t = \ln\left(\frac{1}{2+\epsilon}x \ln x\right)$, with the following lemma; and (ii) letting $\epsilon \downarrow 0$.

Lemma 3.4. *For every $\epsilon > 0$ there exists $a \equiv a(\epsilon) \geq 0$ such that the moment generating function ψ of Z satisfies*

$$\psi(t) \leq \exp[(2 + \epsilon)t^{-1}e^t + at] \quad (3.8)$$

for every $t > 0$.

Granting Lemma 3.4 for the moment, let us follow the outline above to establish Proposition 3.2.

Proof of Proposition 3.2. Choosing $t \equiv t(x) = \ln\left(\frac{1}{2+\epsilon}x \ln x\right)$, for each $\epsilon > 0$ we find

$$\overline{F}(x) \leq e^{-tx}\psi(t) \leq \exp[-tx + (2 + \epsilon)t^{-1}e^t + at].$$

But for fixed $\epsilon > 0$ we have

$$\begin{aligned} e^t &= (2 + \epsilon)^{-1}x \ln x, \\ t &= \ln x + \ln \ln x - \ln(2 + \epsilon), \\ t^{-1} &= (1 + o(1))/\ln x, \end{aligned}$$

so

$$\begin{aligned} &-tx + (2 + \epsilon)t^{-1}e^t + at \\ &= -x \ln x - x \ln \ln x + [\ln(2 + \epsilon)]x + (1 + o(1))x. \end{aligned}$$

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We conclude that

$$\limsup_{x \rightarrow \infty} x^{-1} [\ln \overline{F}(x) - (-x \ln x - x \ln \ln x)] \leq 1 + \ln(2 + \epsilon).$$

Let $\epsilon \downarrow 0$ to complete the proof. □

Inspired by the proof of Lemma 6.1 in [24], which states that there exists $a \geq 0$ such that

$$\psi(t) \leq \exp(e^t + at) \quad \text{for every } t \geq 0,$$

we now establish the right-tail upper bound for ψ given by Lemma 3.4.

Proof of Lemma 3.4. Since $\psi(0) = 1$, it follows by continuity that there exists $t_1 > 0$ such that (3.8) holds [for every $\epsilon > 0$ and any choice of $a(\epsilon) \geq 0$] for $t \in (0, t_1]$. Also, we can choose $\alpha \geq 0$ such that $\psi(t) \leq \exp(\alpha t)$ for $t \in [0, 1]$.

As the proof unfolds, we will see how to choose three parameters

$$a \equiv a(\epsilon) \geq \alpha \text{ sufficiently large,}$$

$$t_2 \equiv t_2(\epsilon) \geq t_1 \text{ sufficiently large,}$$

$$\delta \equiv \delta(\epsilon) > 0 \text{ sufficiently small,}$$

to effect a proof of (3.8).

However t_2 is chosen, we can certainly choose $a \geq \alpha$ so that (3.8) holds for all $t \in [t_1, t_2]$ and hence for $t \in (0, t_2]$. Assume (for the sake of contradiction) that (3.8) fails for some $t > 0$, and let

$$T \equiv T(\epsilon) := \inf\{t > 0 : (3.8) \text{ fails}\}.$$

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Then $T > t_2$, and, by continuity,

$$\psi(T) = \exp[(2 + \epsilon)T^{-1}e^T + aT]. \quad (3.9)$$

We will make frequent use of (3.9) and also use the identity

$$\psi(t) = 2 \int_{u=0}^{1/2} \psi(ut) \psi((1-u)t) e^{tg(u)} du, \quad t \in \mathbb{R}, \quad (3.10)$$

[which follows from (1.2) and symmetry] with $t = T$, together with the simple bound

$$g(u) = 2u \ln u + 2(1-u) \ln(1-u) + 1 \leq 1,$$

to obtain a contradiction by showing that RHS(3.10) for $t = T$ is bounded above by $(1 - \frac{1}{6}\epsilon)\psi(T)$.

For this, break the integral $2 \int_{u=0}^{1/2}$ in (3.10) into “small”, “medium”, and “large” ranges of u :

$$\text{RHS}(3.10) = S + M + L = 2 \int_{u=0}^{\delta/T} + 2 \int_{u=\delta/T}^{1/T} + 2 \int_{u=1/T}^{1/2}$$

with $\delta \equiv \delta(\epsilon) < 1$. To complete the proof, we will show

$$L \leq \frac{1}{12}\epsilon \psi(T), \quad S \leq (1 - \frac{1}{3}\epsilon)\psi(T), \quad M \leq \frac{1}{12}\epsilon \psi(T)$$

when a, t_2, δ are suitably chosen.

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We start with L :

$$\begin{aligned} L &\leq 2e^T \int_{u=1/T}^{1/2} \exp \left[(2+\epsilon) \frac{e^{uT}}{uT} + auT \right. \\ &\quad \left. + (2+\epsilon) \frac{e^{(1-u)T}}{(1-u)T} + a(1-u)T \right] du \\ &\leq 2e^{T+aT} \int_{u=1/T}^{1/2} \exp \left[(2+\epsilon) \left(e^{T/2} + \frac{e^{(1-u)T}}{(1-u)T} \right) \right] du. \end{aligned}$$

We now apply the bound

$$\frac{e^{-uT}}{1-u} \leq 2e^{-1} < 0.8$$

to obtain

$$\begin{aligned} L &\leq 2 \exp[T + aT + (2+\epsilon)e^{T/2}] \times \frac{1}{2} \exp[(2+\epsilon)(0.8)T^{-1}e^T] \\ &= \exp[T + (2+\epsilon)e^{T/2}] \exp[-(0.2)(2+\epsilon)T^{-1}e^T] \\ &\quad \times \exp[(2+\epsilon)T^{-1}e^T + aT] \\ &= \exp[T + (2+\epsilon)e^{T/2}] \exp[-(0.2)(2+\epsilon)T^{-1}e^T] \psi(T) \\ &\leq \exp[(T^2e^{-T} + (2+\epsilon)Te^{-T/2} - 0.4)T^{-1}e^T] \psi(T). \end{aligned}$$

Recall that $T > t_2$. Provided t_2 is chosen sufficiently large, we clearly have

$$L \leq \exp[-(0.3)T^{-1}e^T] \psi(T) \leq \frac{1}{12} \epsilon \psi(T).$$

For the contributions S and M , we begin by observing that the first factor $\psi(uT)$ in the integrand can be bounded above by $\exp(\alpha uT) \leq \exp(auT)$ and the second

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factor by (3.8):

$$\psi((1-u)T) \leq \exp \left[(2+\epsilon) \frac{e^{(1-u)T}}{(1-u)T} + a(1-u)T \right].$$

For S , this gives the bound

$$S \leq 2e^{T+aT} \int_{u=0}^{\delta/T} \exp \left[(2+\epsilon) \frac{e^{(1-u)T}}{(1-u)T} \right] du. \quad (3.11)$$

Observe that the function

$$h(v) := v^{-1}e^v$$

satisfies

$$h'(v) = v^{-2}(v-1)e^v,$$

$$h''(v) = v^{-3}(v^2 - 2v + 2)e^v,$$

$$h'''(v) = v^{-4}(v^3 - 3v^2 + 6v - 6)e^v \geq 0,$$

$$h(v) \leq h(T) - (T-v)h'(T) + \frac{1}{2}(T-v)^2h''(T)$$

where the inequalities hold for $T \geq v \geq 1.6$. Provided $t_2 - \delta \geq 1.6$, we may then

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conclude

$$\begin{aligned}
S &\leq 2e^{T+aT} \times \int_{u=0}^{\delta/T} \exp \left\{ (2+\epsilon)[h(T) - uTh'(T) + \tfrac{1}{2}u^2T^2h''(T)] \right\} du \\
&\leq 2e^{T+aT} \times \int_{u=0}^{\delta/T} \exp \left\{ (2+\epsilon)[h(T) - uTh'(T) + \tfrac{1}{2}\delta uTh''(T)] \right\} du \\
&= 2e^{T+aT} \\
&\quad \times \int_{u=0}^{\delta/T} \exp \left\{ (2+\epsilon)\frac{e^T}{T} \left[1 - (T-1)u + \tfrac{1}{2}\delta T^{-1}((T-1)^2 + 1)u \right] \right\} du \\
&= \psi(T) \\
&\quad \times 2e^T \int_{u=0}^{\delta/T} \exp \left\{ -(2+\epsilon)\frac{e^T}{T} \left[(T-1) - \tfrac{1}{2}\delta T^{-1}((T-1)^2 + 1)u \right] \right\} du \\
&\leq \psi(T) \\
&\quad \times 2e^T \int_{u=0}^{\infty} \exp \left\{ -(2+\epsilon)\frac{e^T}{T} \left[(T-1) - \tfrac{1}{2}\delta T^{-1}((T-1)^2 + 1)u \right] \right\} du \\
&= \psi(T) \times 2e^T \left\{ (2+\epsilon)\frac{e^T}{T} \left[(T-1) - \tfrac{1}{2}\delta T^{-1}((T-1)^2 + 1) \right] \right\}^{-1} \\
&= \psi(T) \times \frac{2}{2+\epsilon} \left[(1 - T^{-1}) - \tfrac{1}{2}\delta T^{-2}((T-1)^2 + 1) \right]^{-1}.
\end{aligned}$$

Without loss of generality $\epsilon < 1$. Provided we choose (independently!) t_2 sufficiently large and δ sufficiently small (in relation to ϵ), we can ensure

$$S \leq (1 - \tfrac{1}{3}\epsilon)\psi(T).$$

The inequality analogous to (3.11) for M is

$$M \leq 2e^{T+aT} \int_{u=\delta/T}^{1/T} \exp \left[(2+\epsilon)\frac{e^{(1-u)T}}{(1-u)T} \right] du.$$

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For M , our bound on $e^{(1-u)T}/[(1-u)T]$ is simpler than for S :

$$\frac{e^{(1-u)T}}{(1-u)T} \leq e^{-\delta} \frac{e^T}{T-1}.$$

The resulting bound on the integrand is then constant in u , and the length of the interval of integration is bounded above by $1/T$. This leads easily to

$$\begin{aligned} M &\leq 2T^{-1} \times \exp \left[- \left\{ 1 - e^{-\delta} \frac{T}{T-1} - \frac{1}{2} T^2 e^{-T} \right\} \times (2 + \epsilon) \frac{e^T}{T} \right] \psi(T) \\ &\leq 2T^{-1} \psi(T) \\ &\leq \frac{1}{12} \epsilon \psi(T), \end{aligned}$$

where the second inequality holds provided t_2 is chosen sufficiently large relative to δ that the expression in $\{\cdot\}$ is nonnegative, and the third inequality holds provided $t_2 \geq 24/\epsilon$.

This completes the proof. □

3.3 Further improvement on right-tail asymptotics

In this section we substantially refine the upper bound of Theorem 3.1(b) with $k = 0$; we also improve the upper bounds for $k \geq 1$, though not as dramatically.

Let

$$J(t) := \int_{s=1}^t \frac{2e^s}{s} ds, \quad t \geq 1. \tag{3.12}$$

It is elementary using integration by parts that $J(t)$ has the (divergent) asymptotic

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expansion

$$J(t) \sim 2t^{-1}e^t \sum_{j=0}^{\infty} j! t^{-j}. \quad (3.13)$$

Here is the main theorem of this section:

Theorem 3.5. *For $x \geq 2e$, let $w \equiv w(x)$ denote the unique real solution satisfying $w \geq 1$ to*

$$x = 2w^{-1}e^w.$$

(a) *As $x \rightarrow \infty$, the limiting QuickSort distribution function F satisfies*

$$\begin{aligned} \overline{F}(x) &\leq \exp[-xw + J(w) - w^2 + O(\log x)] \\ &= \exp[-2e^w + J(w) - w^2 + O(w)]. \end{aligned}$$

(b) *Given an integer $k \geq 1$, as $x \rightarrow \infty$ the k^{th} derivative of the limiting QuickSort distribution function F satisfies*

$$\|\overline{F}^{(k)}\|_x \leq \exp[-xw + J(w) + O(\sqrt{x \log x})]. \quad (3.14)$$

Remark 3.6. (a) We aid the reader in gauging the approximate sizes of the various terms in the bounds appearing in Theorem 3.5. It is routine to check that, as noted by Knessl and Szpankowski [25, eq. (20)],

$$w = \ln(x/2) + \ln \ln(x/2) + (1 + o(1)) \frac{\ln \ln(x/2)}{\ln(x/2)} \quad (3.15)$$

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as $x \rightarrow \infty$. Thus, by (3.13), we have the asymptotic equivalence

$$J(w) \sim 2w^{-1}e^w = x. \quad (3.16)$$

From (3.15)–(3.16) it's easy to see that Theorem 3.5 does indeed strengthen the upper bounds in Theorem 3.1. Inclusion of the term $J(w)$ in the bounds of Theorem 3.5 enables us effectively to bypass the entire infinite asymptotic expansion (3.13).

(b) Using non-rigorous methods, Knessl and Szpankowski [25, see esp. their eq. (18)] derive the following exact asymptotics for $\bar{F}(x)$ as $x \rightarrow \infty$:

$$\bar{F}(x) = \exp \left[-xw + J(w) - w^2 - (\alpha + \tfrac{1}{2})w - \tfrac{3}{2} \ln w + C - \ln(2\sqrt{\pi}) + o(1) \right] \quad (3.17)$$

for some (unspecified) constant C , with $\alpha := 2 \ln 2 + 2\gamma - 1$, where γ denotes the Euler–Mascheroni constant. Hence the bound of Theorem 3.5(a) on $\ln \bar{F}(x)$ matches the conjectured asymptotics to within an additive term $O(w) = O(\log x)$.

(c) In their notation, the non-rigorously derived eq. (88) of [25] should read

$$P(y) \sim \frac{C_*}{\sqrt{2\pi}} \frac{1}{\sqrt{y} w_* \sqrt{1 - (1/w_*)}} \exp \left[-yw_* + \int_1^{w_*} \frac{2e^u}{u} du - w_*^2 - \alpha w_* \right],$$

recalling $\alpha = 2\gamma + 2 \ln 2 - 1$. Ignoring the factor $\sqrt{1 - (1/w_*)}$ which ~ 1 , this result in our notation is

$$\begin{aligned} f(x) &\sim (2\pi \times 2w^{-1}e^w)^{-1/2} e^{-xw} \psi(w) \\ &\sim (2\pi x)^{-1/2} \exp[-xw + J(w) - w^2 - \alpha w - \ln w + C], \end{aligned} \quad (3.18)$$

where ψ is the moment generating function corresponding to f and C is the same

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constant as at (3.17). [They derive their (88) by the “standard saddle point approximation” from the moment generating function expansion (3.20) recalled in Remark 3.8 below, and they derive (3.17) by integrating (3.18).] Hence the bound of Theorem 3.5(b) on $\ln f(x)$ matches the conjectured asymptotics (3.18) to within an additive term $O(\sqrt{x \log x})$.

3.4 Proof of the main Theorem 3.5

In Section 3.4.1 we bound the moment generating function (mgf) ψ of Z . In Section 3.4.2 we prove Theorem 3.5(a) by combining the Chernoff bound

$$\overline{F}(x) = \mathbb{P}(Z \geq x) \leq e^{-tx} \psi(t),$$

for judicious choice of $t \equiv t(x) > 0$, with our bound on ψ . In Section 3.4.3 we prove Theorem 3.5(b).

3.4.1 A bound on the moment generating function of Z

Let ψ denote the mgf of Z . It was shown by Rösler [35] that ψ is everywhere finite. In this subsection we establish a bound on $\psi(t)$ which (for large t) improves on that of Lemma 3.4. Recalling the definition (3.12) of $J(t)$, we next state our bound on $\psi(t)$ which, according to (3.13), does indeed improve on (3.8) for large t .

Proposition 3.7. *There exists a constant $a \geq 0$ such that the moment generating function ψ of Z satisfies*

$$\psi(t) \leq \exp[J(t) - t^2 + at] \tag{3.19}$$

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for every $t \geq 1$.

We postpone the proof of Proposition 3.7 for a preliminary remark.

Remark 3.8. Using non-rigorous methods, Knessl and Szpankowski [25] derive that as $t \rightarrow \infty$ the mgf ψ satisfies

$$\psi(t) = \exp[J(t) - t^2 - \alpha t - \ln t + C + o(1)] \quad (3.20)$$

as $t \rightarrow \infty$ for the same (unspecified) constant C as at (3.17), with $\alpha = 2 \ln 2 + 2\gamma - 1$; see their equation (71) (we have corrected a misplaced-right-parenthesis typo). If (3.20) is true, then our bound on $\ln \psi(t)$ agrees with the truth to within $O(t)$, whereas the bound (3.8) (for fixed ϵ) exceeds the true value by $(1 + o(1))\epsilon t^{-1}e^t$. Thus our bound (3.19) comes substantially closer to the apparent truth than does (3.8).

The proof of Proposition 3.7 will require the following lemma. Recall from Remark 3.8 that $\alpha = 2 \ln 2 + 2\gamma - 1$, and define

$$\widehat{\psi}(t) := \begin{cases} (1 - e^{-t/2}) \exp[J(t) - t^2 - \alpha t - \ln t] & \text{if } t > 1 \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.9. *For all sufficiently large t we have the strict inequality*

$$2 \int_{u=0}^{1/2} \widehat{\psi}(ut) \widehat{\psi}((1-u)t) \exp[tg(u)] \, du < \widehat{\psi}(t).$$

Proof. Call the left side of this inequality $\lambda(t)$. To handle $\lambda(t)$, we begin by changing

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the variable of integration from u to η , where $u = \frac{1}{2}e^{-t}\eta$:

$$\begin{aligned}\lambda(t) &= e^{-t} \int_{\eta=0}^{e^t} \widehat{\psi}\left(\frac{1}{2}te^{-t}\eta\right) \widehat{\psi}\left(t - \frac{1}{2}te^{-t}\eta\right) \exp\left[tg\left(\frac{1}{2}e^{-t}\eta\right)\right] d\eta \\ &= \int_{\eta=0}^{e^t} \widehat{\psi}\left(\frac{1}{2}te^{-t}\eta\right) \widehat{\psi}\left(t - \frac{1}{2}te^{-t}\eta\right) \exp\left[2t\phi\left(\frac{1}{2}e^{-t}\eta\right)\right] d\eta\end{aligned}$$

with $\phi(u) := u \ln u + (1-u) \ln(1-u) \leq 0$.

We next show that the contribution to $\int_{\eta=0}^{e^t}$ here from $\int_{\eta=e^{t/10}}^{e^t}$ is effectively quite negligible. To see this, we consider the integrand in two cases. Before breaking into cases, observe that the second argument for $\widehat{\psi}$ is at least $t/2$, which exceeds 1 if (as we may suppose) $t > 2$. For the first case, suppose that the first argument for $\widehat{\psi}$ also exceeds 1. In this case we need to treat the sum of the J -values at these arguments. But, using the fact that $2s^{-1}e^s$ is monotonically increasing for $s \geq 1$, we see that if $a, b \geq 1$ and $a + b = t$, then

$$\begin{aligned}J(a) + J(b) &= \int_{s=1}^a 2s^{-1}e^s ds + \int_{s=1}^b 2s^{-1}e^s ds \\ &\leq \int_{s=1}^a 2s^{-1}e^s ds + \int_{s=a}^{a+b-1} 2s^{-1}e^s ds = J(t-1)\end{aligned}$$

and therefore

$$\begin{aligned}J(a) + J(b) - J(t) &\leq -[J(t) - J(t-1)] = -\int_{s=t-1}^t 2s^{-1}e^s ds \\ &\leq -2(t-1)^{-1}e^{t-1} = -(1+o(1))2e^{-1}t^{-1}e^t.\end{aligned}$$

For the second case, suppose that the first argument for $\widehat{\psi}$ does not exceed 1. In this

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case we need to treat $J(t - \frac{1}{2}te^{-t}\eta) \leq J(t - \frac{1}{2}te^{-9t/10})$. In this case, observe that

$$\begin{aligned} J(t - \frac{1}{2}te^{-9t/10}) - J(t) &\leq (\frac{1}{2}te^{-9t/10}) \cdot -2(t - \frac{1}{2}te^{-9t/10})^{-1} \exp[t - \frac{1}{2}te^{-9t/10}] \\ &= -(1 + o(1))e^{t/10}. \end{aligned}$$

The minor contribution $\int_{\eta=e^{t/10}}^{e^t}$ is thus bounded between 0 and

$$\begin{aligned} (e^t - e^{t/10}) \times \exp[J(t) - (1 + o(1))e^{t/10} + O(t^2)] \times 1 \\ = \exp[J(t) - (1 + o(1))e^{t/10} + O(t^2)] \\ = \exp[-(1 + o(1))e^{t/10}] \hat{\psi}(t). \end{aligned}$$

For the major contribution $\int_{\eta=0}^{e^{t/10}}$, we can use simple expansions for the first and third factors in the integrand, because $0 \leq \frac{1}{2}te^{-t}\eta \leq \frac{1}{2}te^{-9t/10} = o(1)$:

$$\begin{aligned} \hat{\psi}(\frac{1}{2}te^{-t}\eta) &= 1, \\ \phi(\frac{1}{2}te^{-t}\eta) &= \frac{1}{2}e^{-t}\eta(-t + \ln \eta - \ln 2) - \frac{1}{2}e^{-t}\eta + O(e^{-2t}\eta^2). \end{aligned}$$

We also use an expansion for $J(t - \frac{1}{2}te^{-t}\eta)$ appearing in the second factor in the integrand:

$$\begin{aligned} J(t - \frac{1}{2}te^{-t}\eta) - J(t) &= -\frac{1}{2}te^{-t}\eta J'(t) + \frac{1}{8}t^2e^{-2t}\eta^2 J''(t) + O(t^2e^{-2t}\eta^3) \\ &= -\eta + \frac{1}{4}(t - 1)e^{-t}\eta^2 + O(t^2e^{-2t}\eta^3). \end{aligned}$$

Thus, abbreviating $t - \frac{1}{2}te^{-t}\eta$ as $t_1 \equiv t_1(t, \eta)$, the major contribution to $\lambda(t)$ equals

$$\exp[J(t)]I(t),$$

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where $I(t)$ is the integral

$$I(t) := \int_{\eta=0}^{e^{t/10}} e^{-\eta}(1 - e^{-t_1/2}) \exp \left[\frac{1}{4}(t-1)e^{-t}\eta^2 + O(t^2e^{-2t}\eta^3) \right. \\ \left. + te^{-t}\eta(-t + \ln \eta - \ln 2) - te^{-t}\eta + O(te^{-2t}\eta^2) - t_1^2 - \alpha t_1 - \ln t_1 \right] d\eta.$$

We now use the following additional expansions:

$$t_1^2 = t^2 - t^2e^{-t}\eta + O(t^2e^{-2t}\eta^2), \\ \ln t_1 = \ln(t - \frac{1}{2}te^{-t}\eta) = \ln t - \frac{1}{2}e^{-t}\eta + O(e^{-2t}\eta^2), \\ e^{-t_1/2} = e^{-t/2}[1 + \frac{1}{4}te^{-t}\eta + O(t^2e^{-2t}\eta^2)].$$

Further we can expand the factor $\exp[\cdot]$ appearing in $I(t)$ as $1 + \cdot + O(\cdot^2)$, because $\cdot = o(1)$ uniformly throughout the range of integration.

Calculus now gives

$$I(t) = (1 - e^{-t/2}) \exp[-t^2 - \alpha t - \ln t] \\ \times \left[O(t^4e^{-2t}) + \int_{\eta=0}^{\infty} e^{-\eta}(1 - \frac{1}{4}te^{-3t/2}\eta + \frac{1}{4}(t-1)e^{-t}\eta^2 \right. \\ \left. + te^{-t}\eta(-t + \ln \eta - \ln 2) - te^{-t}\eta + t^2e^{-t}\eta + \frac{1}{2}\alpha te^{-t}\eta + \frac{1}{2}e^{-t}\eta) d\eta \right] \\ = (1 - e^{-t/2}) \exp[-t^2 - \alpha t - \ln t] \times [1 - \frac{1}{4}te^{-3t/2} + O(t^4e^{-2t})].$$

We conclude for sufficiently large t that

$$\lambda(t) = \widehat{\psi}(t)[1 - \frac{1}{4}te^{-3t/2} + O(t^4e^{-2t})] < \widehat{\psi}(t).$$

□

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Remark 3.10. If we change the factor $(1 - e^{-t/2})$ in the definition of $\widehat{\psi}$ to $(1 + e^{-t/2})$, then a similar proof shows that the reverse strict inequality holds in Lemma 3.9. In fact, the proof becomes a bit simpler, since the minor contribution can simply be bounded below by 0.

Proof of Proposition 3.7. We carry out the proof by showing that there exists $a' \geq 0$ such that

$$\psi(t) \leq e^{a't} \widehat{\psi}(t) \quad (3.21)$$

for every $t > 0$.

To begin, we compare asymptotics of $\psi(t)$ and $\widehat{\psi}(t)$ as $t \rightarrow 0$. Because Z has zero mean and finite variance, we have $\psi(t) = 1 + O(t^2)$. On the other hand, $\widehat{\psi}(t) = 1$ for all $0 < t \leq 1$. We can thus choose $t_1 > 0$ and $a'' > 0$ such that (3.21) holds for $t \in [0, t_1]$ and any $a' \geq a''$.

Let $t_2 > 1$ be such that the strict inequality in Lemma 3.9 holds for all $t \geq t_2$, and choose $a' \geq a''$ so that (3.21) holds for $t \in [t_1, t_2]$. Assuming for the sake of contradiction that (3.21) fails for some $t > 0$, let $T := \inf\{t > 0 : (3.21) \text{ fails}\}$. Then $T \geq t_2$, and continuity gives

$$\psi(T) = e^{a'T} \widehat{\psi}(T).$$

Further, if $0 < u < 1$, then (3.21) holds for $t = uT$ and $t = (1 - u)T$, and thus, using our standard integral equation for ψ , we have

$$\psi(T) \leq e^{a'T} \times 2 \int_{u=0}^{1/2} \widehat{\psi}(uT) \widehat{\psi}((1 - u)T) \exp[tg(u)] du,$$

which is strictly smaller than $e^{a'T} \widehat{\psi}(T)$ by applying Lemma 3.9 with $t = T \geq t_2$. The resulting strict inequality $\psi(T) < e^{a'T} \widehat{\psi}(T)$ contradicts the definition of T . Hence (3.21) holds for all $t \geq 0$. \square

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Remark 3.11. Using Remark 3.10 just as Lemma 3.9 is used in the proof of Proposition 3.7, we have the following reverse of Proposition 3.7: There exists a constant $a \geq 0$ such that the mgf ψ of Z satisfies

$$\psi(t) \geq \exp[J(t) - t^2 - at] \quad (3.22)$$

for every $t \geq 1$.

Remark 3.12. (a) Unfortunately, due to the need to handle small values of t in the proofs of Proposition 3.7 and Remark 3.11, we sacrifice the information in the linear term of $\ln \psi(t)$ that Remark 3.8 and Lemma 3.9 strongly suggest. Thus any further progress on asymptotic determination of ψ would have to employ a technique different from the one used to derive [24, Lemma 6.1], (3.8), and (3.19).

(b) The extent to which we are able to make rigorous the claim (3.20) and thereby, in particular, identify the linear term in $\ln \psi(t)$ is the following. If

$$\psi(t) = \exp[J(t) + K(t)]$$

where we *assume* $K'(t) = O(t^{b_1})$ and $K''(t) = O(t^{b_2})$ for some b_1 and b_2 [just as we now know rigorously that $K(t) \sim -t^2 = O(t^2)$], then we must have

$$K(t) = -t^2 - \alpha t - \ln t + C + O(t^b e^{-t})$$

for some constant C , where $b := \max\{4, 2 + 2b_1, 2 + b_2\}$. (Aside: It is natural to *assume further* that $b_1 = 1$ and $b_2 = 0$, in which case $b = 4$.) The proof of this assertion is quite similar to the proof of Lemma 3.9 and is omitted.

3.4.2 Proof of improved asymptotic upper bound on \overline{F}

Proof of Theorem 3.5(a). Choose $t = w$, apply the Chernoff bound

$$\overline{F}(x) = \mathbb{P}(Z \geq x) \leq e^{-tx}\psi(t),$$

and utilize Proposition 3.7 to establish Theorem 3.5(a). □

Remark 3.13. (a) For large x , the optimal choice of t for the Chernoff bound combined with (3.19) is not $t = w$, but rather the larger $\tilde{w} \equiv \tilde{w}(x)$ of the two positive real solutions to

$$x = 2(\tilde{w}^{-1}e^{\tilde{w}} - \tilde{w}) + a.$$

But the resulting improvement in the bound on $\ln \overline{F}(x)$ not only is subsumed by the error bound $O(\log x)$ but in fact is asymptotically equivalent to $2x^{-1}(\log x)^2 = o(1)$ and so is negligible even as concerns estimating $\overline{F}(x)$ to within a factor $1 + o(1)$.

Here is a proof. Use of $t = w$ vs. $t = \tilde{w}$ gives the larger expression

$$-xw + J(w) - w^2 + aw$$

vs.

$$-x\tilde{w} + J(\tilde{w}) - \tilde{w}^2 + a\tilde{w};$$

the increase is

$$\Delta \equiv \Delta(x) := x(\tilde{w} - w) - [J(\tilde{w}) - J(w)] + (\tilde{w}^2 - w^2) - a(\tilde{w} - w).$$

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Using Taylor's theorem, we write

$$\begin{aligned} J(\tilde{w}) - J(w) &= 2w^{-1}e^w(\tilde{w} - w) + t^{-1}e^t(1 - t^{-1})(\tilde{w} - w)^2 \\ &= x(\tilde{w} - w) + (1 + o(1))\frac{1}{2}x(\tilde{w} - w)^2 \end{aligned}$$

where t belongs to (w, \tilde{w}) , and we also note

$$\tilde{w}^2 - w^2 = 2(\tilde{w} - w)(\tilde{w} + w) \sim 2(\tilde{w} - w) \ln x.$$

Thus

$$\Delta = -(1 + o(1))\frac{1}{2}x(\tilde{w} - w)^2 + (1 + o(1))2(\tilde{w} - w) \ln x.$$

It remains to estimate $\tilde{w} - w$. We have

$$1 = \frac{x}{x} = \frac{2(\tilde{w}^{-1}e^{\tilde{w}} - \tilde{w}) + a}{2w^{-1}e^w} = \frac{w}{\tilde{w}}e^{\tilde{w}-w} - \frac{2\tilde{w} - a}{x}.$$

Write this as

$$\frac{w}{\tilde{w}}e^{\tilde{w}-w} = 1 + \frac{2\tilde{w} - a}{x}$$

and take logs. Note

$$\ln \left(\frac{w}{\tilde{w}}e^{\tilde{w}-w} \right) = -\ln \left(1 + \frac{\tilde{w} - w}{w} \right) + \tilde{w} - w \sim \tilde{w} - w$$

and

$$\ln \left(1 + \frac{2\tilde{w} - a}{x} \right) \sim \frac{2\tilde{w} - a}{x}.$$

Thus

$$\tilde{w} - w \sim 2x^{-1} \ln x.$$

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It now follows that

$$\Delta = -(1 + o(1))2x^{-1}(\ln x)^2 + (1 + o(1))4x^{-1}(\ln x)^2 \sim 2x^{-1}(\ln x)^2,$$

as claimed.

(b) If we grant the truth of (3.20), the following upper bound on $\overline{F}(x)$ resulting from use of a Chernoff inequality with $t = w$ together with (3.20) still does not completely match (3.17):

$$\begin{aligned} \overline{F}(x) &\leq \exp[-xw + J(w) - w^2 - \alpha w - \ln w + C + o(1)] \\ &= 2\sqrt{\pi} w^{1/2} e^{w/2} \times \text{RHS}(3.17) \sim (2\pi x)^{1/2} \times \text{RHS}(3.17). \end{aligned}$$

Further, use of the *exactly* optimal t [ignoring the $o(1)$ remainder term in (3.20)] gives a bound that is still asymptotically $(2\pi x)^{1/2} \times \text{RHS}(3.17)$. Thus if the asymptotic inequality $\overline{F}(x) \leq \text{RHS}(3.17)$ is ever to be established rigorously, it would have to involve some technique (such as a rigorization of the saddle-point arguments used in [25]) we have not used; Chernoff bounds are insufficient.

3.4.3 Proof of improved asymptotic upper bounds on absolute values of derivatives of F

Using the improved right-tail upper bound of the distribution function in Theorem 3.5(a), we are now able to establish Theorem 3.5(b).

Proof of Theorem 3.5(b). The bound (3.14) holds for $k = 0$ because it is cruder than the bound of Theorem 3.5(a). The bound (3.14) for general values of k then follows

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inductively using Proposition 2.9, according to which

$$\limsup_{x \rightarrow \infty} r(x)^{-1} \left(\ln \|\overline{F}^{(k+1)}\|_x - \ln \|\overline{F}^{(k)}\|_x \right) \leq 0$$

provided $r(x) = \omega(\sqrt{x \log x})$ as $x \rightarrow \infty$. □

3.5 Large deviations for QuickSort

McDiarmid and Hayward [31] study large deviations for the variant of **QuickSort** in which the pivot (that is, the initial partitioning key) is chosen as the median of $2t+1$ keys chosen uniformly at random without replacement from among all the keys. The case $t = 0$ is the classical **QuickSort** algorithm of our ongoing limited focus in this paper. Restated equivalently in terms of the random variable Z_n in (1.1) (as straightforward calculation reveals), the following is their main theorem for classical **QuickSort**.

Theorem 3.14 ([31]). *Let x_n satisfy*

$$\frac{\mu_n}{n \ln n} < x_n \leq \frac{\mu_n}{n}. \quad (3.23)$$

Then as $n \rightarrow \infty$ we have

$$\mathbb{P}(|Z_n| > x_n) = \exp\{-x_n[\ln x_n + O(\log \log \log n)]\}. \quad (3.24)$$

Observe that (3.23) is roughly equivalent to the condition that x_n lie between 2 and $2 \ln n$, and rather trivially the range can be extended to $1 < x_n \leq \mu_n/n$. But notice also that if $x_n = (\ln \ln n)^{c_n}$ with c_n nondecreasing (say), then (3.24) provides

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a nontrivial upper bound on $\mathbb{P}(|Z_n| > x_n)$ if and only if $c_n \rightarrow \infty$.

McDiarmid and Hayward require a fairly involved proof utilizing primarily the method of bounded differences pioneered by McDiarmid [32] to establish the \leq half of (3.24). The \geq half is proven by establishing (by means of another substantial argument) the right-tail lower bound

$$\mathbb{P}(Z_n > x_n) \geq \exp\{-x_n[\ln x_n + O(\log \log \log n)]\}, \quad (3.25)$$

again assuming (3.23) (see [31, Lemma 2.9]). It follows from (3.24)–(3.25) that we have the right-tail large deviation result that

$$\mathbb{P}(Z_n > x_n) = \exp\{-x_n[\ln x_n + O(\log \log \log n)]\}. \quad (3.26)$$

The main point of this section [see Theorem 3.16(b)–(d)] is to note that (3.26) can be refined, for deviations not allowed to be quite as large as those permitted by Theorem 3.14, rather effortlessly by combining our upper bound [Theorem 3.5(a)] and lower bound [Theorem 3.1(b), with $k = 0$] on the right tail of F with the following bound on Kolmogorov–Smirnov distance between the distributions of Z_n and Z (see [16, Section 5]):

Lemma 3.15 ([16]). *We have*

$$\sup_x |\mathbb{P}(Z_n > x) - \mathbb{P}(Z > x)| \leq \exp \left[-\frac{1}{2} \ln n + O((\log n)^{1/2}) \right].$$

We state next our right-tail large-deviations theorem for **QuickSort**. With the additional indicated restriction on the growth of x_n (which allows for x_n nearly as large as $\frac{1}{2} \frac{\ln n}{\ln \ln n}$), parts (b)–(c) strictly refine (3.25) and the asymptotic upper bound

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on $\mathbb{P}(Z_n > x_n)$ implied by (3.26). The left-hand endpoint of the interval I_n in Theorem 3.16 is chosen as $c > 1$ simply to ensure that $\sup\{-\ln \ln x : x \in I_n\} < \infty$.

Theorem 3.16. *Let (ω_n) be any sequence diverging to $+\infty$ as $n \rightarrow \infty$ and let $c > 1$.*

For integer $n \geq 3$, consider the interval $I_n := [c, \frac{1}{2} \frac{\ln n}{\ln \ln n} (1 - \frac{\omega_n}{\ln \ln n})]$.

(a) *Uniformly for $x \in I_n$ we have*

$$\mathbb{P}(Z_n > x) = (1 + o(1))\mathbb{P}(Z > x) \quad \text{as } n \rightarrow \infty. \quad (3.27)$$

(b) *If $x_n \in I_n$ for all large n , then*

$$\mathbb{P}(Z_n > x_n) \geq \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)]. \quad (3.28)$$

(c) *If $x_n \in I_n$ for all large n and $x_n \rightarrow \infty$, then*

$$\mathbb{P}(Z_n > x_n) \leq \exp[-x_n w_n + J(w_n) - w_n^2 + O(\log x_n)] \quad (3.29)$$

$$= \exp[-x_n \ln x_n - x_n \ln \ln x_n + (1 + \ln 2)x_n + o(x_n)], \quad (3.30)$$

where w_n is the larger of the two real solutions to $x_n = 2w_n^{-1}e^{w_n}$.

(d) *If $x_n \in I_n$ for all large n , then*

$$\mathbb{P}(Z_n > x_n) = \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)]. \quad (3.31)$$

Proof. Parts (b)–(c) follow immediately from part (a) and Theorem 3.5(a), and part (d) by combining parts (b)–(c). So we need only prove part (a), for which by Lemma 3.15 it is sufficient to prove that

$$\exp\left[-\frac{1}{2} \ln n + O((\log n)^{1/2})\right] \leq o(\mathbb{P}(Z > x_n))$$

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with $x_n \equiv \frac{1}{2} \frac{\ln n}{\ln \ln n} \left(1 - \frac{\omega_n}{\ln \ln n}\right)$; this assertion decreases in strength as the choice of ω_n is increased, so we may assume that $\omega_n = o(\log \log n)$. Since, by Theorem 3.1(b), we have

$$\mathbb{P}(Z > x_n) \geq \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)],$$

it suffices to show that for any constant $C < \infty$ we have

$$-\frac{1}{2} \ln n + C(\ln n)^{1/2} + x_n \ln x_n + x_n \ln \ln x_n + Cx_n \rightarrow -\infty.$$

But, writing L for \ln and L_k for the k th iterate of L , and abbreviating $\alpha_n := 1 - \frac{\omega_n}{L_2 n}$, this follows from the observation that, for n large,

$$\begin{aligned} & x_n(L x_n + L_2 x_n + C) \\ &= \frac{1}{2} \frac{L n}{L_2 n} \alpha_n [(L_2 n - L_3 n - L 2 + L \alpha_n) + L(L_2 n - L_3 n - L 2 + L \alpha_n) + C] \\ &= \frac{1}{2} \frac{L n}{L_2 n} \alpha_n \left[L_2 n + C - L 2 + L \alpha_n + L \left(1 - \frac{L_3 n + L 2 - L \alpha_n}{L_2 n} \right) \right] \\ &= \frac{1}{2} \frac{L n}{L_2 n} \alpha_n \left[L_2 n + C - L 2 + L \alpha_n - (1 + o(1)) \frac{L_3 n}{L_2 n} \right] \\ &= \frac{1}{2} \frac{L n}{L_2 n} \alpha_n [L_2 n + C - L 2 + o(1)] \\ &= \left(\frac{1}{2} L n \right) \alpha_n \left[1 + \frac{C - L 2 + o(1)}{L_2 n} \right] = \frac{1}{2} L n - (1 + o(1)) \omega_n \frac{L n}{2 L_2 n}. \end{aligned}$$

□

For completeness we next present a left-tail analogue of Theorem 3.16 [but, for brevity, only parts (b)–(c) thereof]. Theorem 3.17 follows in similar fashion using the case $k = 0$ of (3.7) in place of Theorem 3.1(b). No such left-tail large-deviation result is found in [31]. Recall $\Gamma := (2 - \frac{1}{\ln 2})^{-1}$ and the notation L_k used in the proof of Theorem 3.16.

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Theorem 3.17. *If $1 < x_n \leq \Gamma^{-1}(\mathsf{L}_2 n - \mathsf{L}_4 n - \omega_n)$ with $\omega_n \rightarrow \infty$, then*

$$\exp \left[-e^{\Gamma x_n + \mathsf{L}_2 x_n + O(1)} \right] \leq \mathbb{P}(Z_n \leq -x_n) \leq \left[-e^{\Gamma x_n + O(1)} \right].$$

Remark 3.18. The upper bound in Theorem 3.17 requires only the weaker restriction

$$-M \leq x_n \leq \Gamma^{-1}(\mathsf{L}_2 n - \omega_n)$$

with $M < \infty$ and $\omega_n \rightarrow \infty$.

Remark 3.19. If we let $N := n + 1$ and study the slight modification $\widehat{Z}_n := (X_n - \mu_n)/N = \lceil n/(n+1) \rceil Z_n$ instead of (1.1), then large deviation upper bounds based on tail estimates of the limiting F have broader applicability and are easier to derive, too. The reason is that (i) both Theorem 3.5(a) and the upper bound for $k = 0$ in (3.7) have been derived by establishing an upper bound on the limiting mgf ψ and using a Chernoff bound, and (ii) according to [16, Theorem 7.1], ψ majorizes the moment generating function $\widehat{\psi}_n$ of \widehat{Z}_n for every n . It follows immediately (with w defined in the now-familiar way in terms of x) that $\mathbb{P}(\widehat{Z}_n > x)$ (respectively, $\mathbb{P}(\widehat{Z}_n \leq -x)$) is bounded above uniformly in n by

$$\exp[-xw + J(w) - w^2 + O(\log x)] \tag{3.32}$$

$$= \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)] \tag{3.33}$$

(resp., by $\exp \left[-e^{\gamma x + O(1)} \right]$) as $x \rightarrow \infty$; there is *no restriction at all* on how large x can be in terms of n .

Here are examples of *very* large values of x for which the tail probabilities are nonzero and the aforementioned bounds still match logarithmic asymptotics to lead

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order of magnitude, albeit not to lead-order term. Let \lg denote binary log. The largest possible value of X_n is $\binom{n}{2}$ (corresponding to any binary search tree which is a path), which occurs with probability $2^{n-1}/n!$. The smallest possible value (supposing, for simplicity, that $n = 2^k - 1$ for integer k) is $(k - 2)2^k + 2 = N(\lg N - 2) + 2$ (corresponding to the perfect tree, in the terminology of [7, Section 3]); according to [7, Proposition 4.1], this value occurs with probability $\exp[-s(1)N + s(N + 1)]$, where

$$s(\nu) := \sum_{j=1}^{\infty} 2^{-j} \ln(2^j \nu - 1).$$

Correspondingly, the largest possible value of \widehat{Z}_n is

$$\lambda_n := \frac{n(n+7)}{2(n+1)} - 2H_n = (1 + o(1))\frac{1}{2}N,$$

and the smallest is $-\sigma_n$, with

$$\sigma_n := -2H_N - \lg N - 2 = (2 - \frac{1}{\ln 2}) \ln N + O(1).$$

The bound (3.33) on $\mathbb{P}(\widehat{Z}_n > \lambda_n)$ is in fact also (by the same proof) a bound on the larger probability $\mathbb{P}(\widehat{Z}_n \geq \lambda_n)$, and equals

$$\exp \left\{ -\frac{1}{2}N[\ln N + \ln \ln N - (2 \ln 2 + 1) + o(1)] \right\},$$

whereas (using Stirling's formula) the truth is

$$\mathbb{P}(\widehat{Z}_n \geq \lambda_n) = \exp[-N \ln N + (1 + \ln 2)N + O(\log N)].$$

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The bound on $\mathbb{P}(\hat{Z}_n \leq -\sigma_n)$ equals

$$\exp[-e^{\ln N + O(1)}] = \exp[-\Omega(N)],$$

whereas (by [7, Proposition 4.1 and Table 1]) the truth is

$$\mathbb{P}(\hat{Z}_n \leq -\sigma_n) = \exp[-s(1)N + O(\log N)]$$

and (rounded to seven decimal places) $s(1) = 0.9457553$.

Chapter 4

Limiting QuickQuant distribution: Introduction and existence of density function f_t

4.1 Introduction

`QuickQuant` is closely related to an algorithm called `QuickSelect`, which in turn can be viewed as a one-sided analogue of `QuickSort`. In brief, `QuickSelect`(n, m) is an algorithm designed to find a number of rank m in an unsorted list of size n . It works by recursively applying the same partitioning step as `QuickSort` to the sublist that contains the item of rank m until the pivot we pick *has* the desired rank or the size of the sublist to be explored has size one. Let $C_{n,m}$ denote the number of comparisons needed by `QuickSelect`(n, m). Knuth [26] finds the formula

$$\mathbb{E} C_{n,m} = 2[(n+1)H_n - (n+3-m)H_{n+1-m} - (m+2)H_m + (n+3)] \quad (4.1)$$

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for the expectation.

The algorithm **QuickQuant**(n, t) refers to **QuickSelect**(n, m_n) such that the ratio m_n/n converges to a specified value $t \in [0, 1]$ as $n \rightarrow \infty$. It is easy to see that (4.1) tells us about the limiting behavior of the expected number of comparisons after standardizing:

$$\lim_{n \rightarrow \infty} \mathbb{E}[n^{-1}C_{n,m_n}] = 2 + 2H(t), \quad (4.2)$$

where $H(x) := -x \ln x - (1-x) \ln(1-x)$ with $0 \ln 0 := 0$.

We follow the set-up and notation of Fill and Nakama [18], who use an infinite sequence $(U_i)_{i \geq 1}$ of independent $\text{Uniform}(0, 1)$ -distributed random variables to couple the number of key comparisons C_{n,m_n} for all n . Let $L_0(n) := 0$ and $R_0(n) := 1$. For $k \geq 1$, inductively define

$$\tau_k(n) := \inf\{i \leq n : L_{k-1}(n) < U_i < R_{k-1}(n)\},$$

and let $r_k(n)$ be the rank of the pivot $U_{\tau_k(n)}$ in the set $\{U_1, \dots, U_n\}$ if $\tau_k(n) < \infty$ and be m_n otherwise. [Recall that the infimum of the empty set is ∞ ; hence $\tau_k(n) = \infty$ if and only if $L_{k-1}(n) = R_{k-1}(n)$.] Also, inductively define

$$L_k(n) := \mathbb{1}(r_k(n) \leq m_n)U_{\tau_k(n)} + \mathbb{1}(r_k(n) > m_n)L_{k-1}(n), \quad (4.3)$$

$$R_k(n) := \mathbb{1}(r_k(n) \geq m_n)U_{\tau_k(n)} + \mathbb{1}(r_k(n) < m_n)R_{k-1}(n), \quad (4.4)$$

if $\tau_k(n) < \infty$, but

$$(L_k(n), R_k(n)) := (L_{k-1}(n), R_{k-1}(n)).$$

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if $\tau_k(n) = \infty$. The number of comparisons at the k^{th} step is then

$$S_{n,k} := \sum_{i: \tau_k < i \leq n} \mathbb{1}(L_{k-1}(n) < U_i < R_{k-1}(n)),$$

and the normalized total number of comparisons equals

$$n^{-1}C_{n,m_n} := n^{-1} \sum_{k \geq 1} S_{n,k}. \quad (4.5)$$

Mahmoud, Modarres and Smythe [29] studied **QuickSelect** in the case that the rank m is taken to be a random variable M_n uniformly distributed on $\{1, \dots, n\}$ and assumed to be independent of the numbers in the list. They used the Wasserstein metric to prove that $Z_n := n^{-1}C_{n,M_n} \xrightarrow{\mathcal{L}} Y$ as $n \rightarrow \infty$ and identified the distribution of Y . In particular, they proved that Y has an absolutely continuous distribution function. Grübel and Rösler [21] treated all the quantiles t simultaneously by letting $m_n \equiv m_n(t)$. Specifically, they considered the normalized process X_n defined by

$$X_n(t) := n^{-1}C_{n, \lfloor nt \rfloor + 1} \text{ for } 0 \leq t < 1, \quad X_n(t) := n^{-1}C_{n,n} \text{ for } t = 1. \quad (4.6)$$

Working in the Skorohod topology (see Billingsley [2, Chapter 3]), they proved that this process has a limiting distribution as $n \rightarrow \infty$, and the value of the limiting process at argument t is the sum of the lengths of all the intervals encountered in all the steps of searching for population quantile t . We can use the *same* sequence $(U_i)_{i \geq 1}$ of Uniform(0, 1) random variables to express the limiting stochastic process.

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Let $L_0(t) := 0$ and $R_0(t) := 1$. For $k \geq 1$, inductively define

$$\tau_k := \inf\{i : L_{k-1}(t) < U_i < R_{k-1}(t)\}, \quad (4.7)$$

$$L_k(t) := \mathbb{1}(U_{\tau_k(t)} < t) U_{\tau_k(t)} + \mathbb{1}(U_{\tau_k(t)} > t) L_{k-1}(t), \quad (4.8)$$

$$R_k(t) := \mathbb{1}(U_{\tau_k(t)} < t) R_{k-1}(t) + \mathbb{1}(U_{\tau_k(t)} > t) U_{\tau_k(t)}. \quad (4.9)$$

The limiting process Z can be expressed as

$$Z(t) := \sum_{k=0}^{\infty} [R_k(t) - L_k(t)] = 1 + \sum_{k=1}^{\infty} [R_k(t) - L_k(t)]. \quad (4.10)$$

Grübel and Rösler [21, Theorem 8] also proved that we can replace the subscript $\lfloor nt \rfloor + 1$ in (4.6) by any $m_n(t)$ with $0 \leq m_n(t) \leq n$ such that $m_n(t)/n \rightarrow t$ as $n \rightarrow \infty$, and then the normalized random variables $n^{-1}C_{n,m_n(t)}$ converge (univariately) to the limiting random variable $Z(t)$ for each $t \in [0, 1]$.

Kodaj and Móri [27] proved the (univariate) convergence of (4.5) to $Z(t)$ in the Wasserstein metric. Using the coupling technique and induction, they proved that (4.5) is stochastically smaller than its continuous counterpart (4.10). Combining this fact with knowledge of their expectations (see (4.1) and [27, Lemma 2.2]), they proved that (4.5) converges to (4.10) in the Wasserstein metric and thus in distribution.

Grübel [20] connected **QuickSelect**(n, m_n) to a Markov chain to identify the limiting process. For each fixed $n \geq 1$, he considered the Markov chain $(Y_m^{(n)})_{m \geq 0}$ on the state space $I_n := \{(i, j) : 1 \leq j \leq i \leq n\}$ with $Y_0^{(n)} := (n, m_n)$. Transition probabilities of $Y^{(n)}$ from the state (i, j) are determined by the partition step of **QuickSelect**(i, j) as follows. If $Y_m^{(n)} = (i, j)$, then $Y_{m+1}^{(n)}$ is selected uniformly at

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random from the set

$$\{(i - k, j - k) : k = 1, \dots, j - 1\} \cup \{(1, 1)\} \cup \{(i - k, j) : k = 1, \dots, i - j\};$$

in particular, $(1, 1)$ is an absorbing state for $Y^{(n)}$. If we write $Y_m^{(n)} = (S_m^{(n)}, Q_m^{(n)})$, then we know

$$n^{-1}C_{n,m_n} \stackrel{\mathcal{L}}{=} n^{-1} \sum_{m \geq 0} (S_m^{(n)} - 1). \quad (4.11)$$

Grübel [20] constructed another Markov chain $Y = (Y_m) = ((S_m, Q_m))$, which is a continuous-value counterpart of the process $Y^{(n)}$, and he proved that for all $m \geq 0$, the random vector $Y_m^{(n)}$ converges to Y_m almost surely. Using the dominated convergence theorem, he proved that the random variables $n^{-1} \sum_{m=0}^{\infty} (S_m^{(n)} - 1)$ converge almost surely to $\sum_{m=0}^{\infty} S_m$; the limiting random variable here is exactly $Z(t)$ of (4.10). Combining with (4.25), he concluded that $n^{-1}C_{n,m_n}$ converges in distribution to (4.10). Hwang and Tsai [23] identified the limiting distribution of (4.5) when $m_n = o(n)$ as the Dickman distribution.

Fill and Nakama [18] studied the limiting distribution of the cost of using **QuickSelect** for a variety of cost functions. In particular, when there is simply *unit* cost of comparing any two keys, then their work reduces to study of the number of key comparisons, to which we limit our focus here. They proved L^p -convergence of (4.5) for **QuickQuant** (n, t) to (4.10) for $1 \leq p < \infty$ by first studying the distribution of the number of key comparisons needed for another algorithm called **QuickVal**, and then comparing the two algorithms. The algorithm **QuickVal** (n, t) finds the rank of the *population* t -quantile in the sample, while its cousin **QuickQuant** (n, t) looks for the sample t -quantile. Intuitively, when the sample size is large, we expect the rank of the population t -quantile to be close to nt . Therefore, the two algorithms should

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behave similarly when n is large. Given a set of keys $\{U_1, \dots, U_n\}$, where U_i are i.i.d. $\text{Uniform}(0, 1)$ random variables, one can regard the operation of $\text{QuickVal}(n, t)$ as that of finding the rank of the value t in the augmented set $\{U_1, \dots, U_n, t\}$. It works by first selecting a pivot uniformly at random from the set of keys $\{U_1, \dots, U_n\}$ and then using the pivot to partition the *augmented* set (we don't count the comparison of the pivot with t). We then recursively do the same partitioning step on the subset that contains t until the set of the keys on which the algorithm operates reduces to the singleton $\{t\}$. For $\text{QuickVal}(n, t)$ with the definitions (4.7)–(4.9) and with

$$S_{n,k}(t) := \sum_{\tau_k(t) < i \leq n} \mathbb{1}(L_{k-1}(t) < U_i < R_{k-1}(t)),$$

Fill and Nakama [18] showed that $n^{-1} \sum_{k \geq 1} S_{n,k}(t)$ converges (for fixed t) almost surely and also in L^p for $1 \leq p < \infty$ to (4.10). They then used these facts to prove the L^p convergence (for fixed t) of (4.5) to (4.10) for $\text{QuickQuant}(n, t)$ for $1 \leq p < \infty$.

Fill and Matterer [17] treated distributional convergence for the worst-case cost of **Find** for a variety of cost functions. Suppose, for example, that we continue, as at the start of this section, to assign unit cost to the comparison of any two keys, so that $C_{n,m}$ is the total cost for $\text{QuickSelect}(n, m)$. Then (for a list of length n) the cost of worst-case **Find** is $\max_{1 \leq m \leq n} C_{n,m}$, and its distribution depends on the joint distribution of $C_{n,m}$ for varying m . We shall not be concerned here with worst-case **Find**, but we wish to review the approach and some of the results of [17], since there is relevance of their work to $\text{QuickQuant}(n, t)$ for fixed t .

Fill and Matterer [17] considered tree-indexed processes closely related to the operation of the **QuickSelect** algorithm, as we now describe. For each node in a given rooted ordered binary tree, let θ denote the binary sequence (or *string*) representing

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the path from the root to this node, where 0 corresponds to taking the left child and 1 to taking the right. The value of θ for the root is thus the empty string, denoted ε . Define $L_\varepsilon := 0$, $R_\varepsilon := 1$, and $\tau_\varepsilon := 1$. Given a sequence of i.i.d. $\text{Uniform}(0, 1)$ random variables U_1, U_2, \dots , recursively define

$$\begin{aligned}\tau_\theta &:= \inf\{i : L_\theta < U_i < R_\theta\}, \\ L_{\theta 0} &:= L_\theta, \quad L_{\theta 1} := U_{\tau_\theta}, \\ R_{\theta 0} &:= U_{\tau_\theta}, \quad R_{\theta 1} := R_\theta.\end{aligned}$$

Here the concatenated string $\theta 0$ corresponds to the left child of the node with string θ , while $\theta 1$ corresponds to the right child. Observe that, when inserting a key U_i arriving at time $i > \tau_\theta$ into the binary tree, this key is compared with the “pivot” U_{τ_θ} if and only if $U_i \in (L_\theta, R_\theta)$. For n insertions, the total cost of comparing keys with pivot U_{τ_θ} is therefore

$$S_{n,\theta} := \sum_{\tau_\theta < i \leq n} \mathbb{1}(L_\theta < U_i < R_\theta).$$

We define a binary-tree-indexed stochastic process $S_n = (S_{n,\theta})_{\theta \in \Theta}$, where Θ is the collection of all finite-length binary sequences.

For each $1 \leq p \leq \infty$, Fill and Matterer [17, Definition 3.10 and Proposition 3.11] defined a Banach space $\mathcal{B}^{(p)}$ of binary-tree-indexed stochastic processes that corresponds in a natural way to the Banach space L^p for random variables. Let $I_\theta := R_\theta - L_\theta$ and consider the process $I = (I_\theta)_{\theta \in \Theta}$. Fill and Matterer [17, Theorem 4.1 with $\beta \equiv 1$] proved the convergence of the processes $n^{-1}S_n$ to I in the Banach space $\mathcal{B}^{(p)}$ for each $2 \leq p < \infty$.

For the simplest application in [17], namely, to **QuickVal**(n, t) with t fixed, let $\gamma(t)$ be the infinite path from the root to the key having value t in the (almost

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surely) complete binary search tree formed by successive insertions of U_1, U_2, \dots into an initially empty tree. The total cost (call it V_n) of **QuickVal**(n, t) can then be computed by summing the cost of comparisons with each (pivot-)node along the path, that is,

$$V_n := \sum_{\theta \in \gamma(t)} S_{n,\theta}.$$

Using their tree-process convergence theorem described in our preceding paragraph, Fill and Matterer [17, Proposition 6.1 with $\beta \equiv 1$] established L^p -convergence, for each $0 < p < \infty$, of $n^{-1}V_n$ to $I_{\gamma(t)}$ as $n \rightarrow \infty$, where $I_{\gamma(t)} := \sum_{\theta \in \gamma(t)} I_\theta$. Moreover ([17, Theorem 6.3 with $\beta \equiv 1$]), they also proved L^p -convergence of $n^{-1}Q_n$ to the same limit, again for every $0 < p < \infty$, where Q_n denotes the cost of **QuickQuant**(n, t).

Throughout the remainder of this dissertation, we will use the standard notations $\mathbf{1}(A)$ to denote the indicator function of the event A and $\mathbb{E}[f; A]$ for $\mathbb{E}[f\mathbf{1}(A)]$.

4.1.1 Summary

In Section 4.2, by construction we establish the existence of densities f_t for the random variables $Z(t)$ defined in (4.10). In Section 5.1 we prove that these densities are uniformly bounded and in Section 5.2 that they are uniformly continuous. As shown in Section 6.1, the densities satisfy a certain integral equation for $0 < t < 1$. The right-tail behavior of the density functions is examined in Section 6.3, and the left-tail behavior in Section 6.4. In Section 6.2 we prove that $f_t(x)$ is positive if and only if $x > \min\{t, 1 - t\}$, and we improve the result of Section 5.2 by showing that $f_t(x)$ is Lipschitz continuous in x for fixed t in Section 7.1 and jointly continuous in (t, x) in Section 7.2. Sections 8.1–8.2 are devoted to sharp logarithmic asymptotics for the right tail of f_t , and Section 8.3 uses the results of those two sections to treat

right-tail large deviation behavior of $\text{QuickQuant}(n, t)$ for large but finite n .

4.2 Existence (and construction) of density functions

In this section, we prove that $Z \equiv Z(t)$ defined in (4.10) for fixed $0 \leq t \leq 1$ has a density. For notational simplification, we let $L_k \equiv L_k(t)$ and $R_k \equiv R_k(t)$. Let $J \equiv J(t) := Z(t) - 1 = \sum_{k=1}^{\infty} \Delta_k$ with $\Delta_k \equiv \Delta_k(t) := R_k(t) - L_k(t)$. We use convolution notation as in Section V.4 of Feller [9]. The following lemma is well known and can be found, for example, in Feller[9, Theorem V.4.4] or Durrett[8, Theorem 2.1.11].

Lemma 4.1. *If X and Y are independent random variables with respective distribution functions F and G , then $Z = X + Y$ has the distribution function $F \star G$. If, in addition, X has a density f (with respect to Lebesgue measure), then Z has a density $f \star G$.*

Let $X = \Delta_1 + \Delta_2$ and $Y = \sum_{k=3}^{\infty} \Delta_k$. If we condition on $(L_3, R_3) = (l_3, r_3)$ for some $0 \leq l_3 < t < r_3 \leq 1$ with $(l_3, r_3) \neq (0, 1)$, we then have

$$Y = (r_3 - l_3) \sum_{k=3}^{\infty} \frac{R_k - L_k}{r_3 - l_3} = (r_3 - l_3) \sum_{k=3}^{\infty} (R'_k - L'_k), \quad (4.12)$$

where we set $L'_k = (L_k - l_3)/(r_3 - l_3)$ and $R'_k = (R_k - l_3)/(r_3 - l_3)$ for $k \geq 3$. Observe that, by definitions (4.7)–(4.9), the stochastic process $((L'_k, R'_k))_{k \geq 3}$, conditionally given $(L_3, R_3) = (l_3, r_3)$, has the same distribution as the (unconditional) stochastic process of intervals $((L_k, R_k))_{k \geq 0}$ encountered in all the steps of searching for population quantile $(t - l_3)/(r_3 - l_3)$ (rather than t) by QuickQuant . Note also

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that (again conditionally) the stochastic processes $((L_k, R_k))_{0 \leq k \leq 2}$ and $((L'_k, R'_k))_{k \geq 3}$ are independent. Thus (again conditionally) $Y/(r_3 - l_3)$ has the same distribution as the (unconditional) random variable $Z((t - l_3)/(r_3 - l_3))$ and is independent of X . We will prove later (Lemmas 4.3–4.4) that, conditionally given $(L_3, R_3) = (l_3, r_3)$, the random variable X has a density. Let

$$f_{l_3, r_3}(x) := \mathbb{P}(X \in dx \mid (L_3, R_3) = (l_3, r_3))/dx$$

be such a conditional density. We can then use Lemma 4.1 to conclude that $J = X + Y$ has a conditional density

$$h_{l_3, r_3}(x) := \mathbb{P}(J \in dx \mid (L_3, R_3) = (l_3, r_3))/dx.$$

By mixing $h_{l_3, r_3}(x)$ for all possible values of l_3, r_3 , we will obtain an unconditional density function for J , as summarized in the following theorem.

Theorem 4.2. *For each $0 \leq t \leq 1$, the random variable $J(t) = Z(t) - 1$ has a density*

$$f_t(x) := \int \mathbb{P}((L_3, R_3) \in d(l_3, r_3)) \cdot h_{l_3, r_3}(x), \quad (4.13)$$

and hence the random variable $Z(t)$ has density $f_t(x - 1)$.

Now, as promised, we prove that, conditionally given $(L_3, R_3) = (l_3, r_3)$, the random variable X has a density f_{l_3, r_3} . We begin with the case $0 < t < 1$.

Lemma 4.3. *Let $0 \leq l_3 < t < r_3 \leq 1$ with $(l_3, r_3) \neq (0, 1)$. Conditionally given $(L_3, R_3) = (l_3, r_3)$, the random variable $X = \Delta_1 + \Delta_2$ has a right continuous density f_{l_3, r_3} .*

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Proof. We consider three cases based on the values of (l_3, r_3) .

Case 1: $l_3 = 0$ and $r_3 < 1$. Since L_k is nondecreasing in k , from $L_3 = 0$ it follows that $L_1 = L_2 = 0$. The unconditional joint distribution of $(L_1, R_1, L_2, R_2, L_3, R_3)$ satisfies

$$\begin{aligned} \mathbb{P}(L_1 = 0, R_1 \in dr_1, L_2 = 0, R_2 \in dr_2, L_3 = 0, R_3 \in dr_3) \\ = \mathbb{1}(t < r_3 < r_2 < r_1 < 1) dr_1 \frac{dr_2}{r_1} \frac{dr_3}{r_2} \end{aligned} \quad (4.14)$$

and hence

$$\begin{aligned} \mathbb{P}(L_3 = 0, R_3 \in dr_3) &= \mathbb{1}(t < r_3 < 1) dr_3 \int_{r_2=r_3}^1 \frac{dr_2}{r_2} \int_{r_1=r_2}^1 \frac{dr_1}{r_1} \\ &= \mathbb{1}(t < r_3 < 1) dr_3 \int_{r_2=r_3}^1 \frac{dr_2}{r_2} (-\ln r_2) \\ &= \frac{1}{2} (\ln r_3)^2 \mathbb{1}(t < r_3 < 1) dr_3. \end{aligned} \quad (4.15)$$

Dividing (4.14) by (4.15), we find

$$\begin{aligned} \mathbb{P}(L_1 = 0, R_1 \in dr_1, L_2 = 0, R_2 \in dr_2 \mid L_3 = 0, R_3 = r_3) \\ = \frac{2 r_1^{-1} r_2^{-1} dr_1 dr_2}{(\ln r_3)^2} \mathbb{1}(t < r_3 < r_2 < r_1 < 1). \end{aligned} \quad (4.16)$$

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Thus for $x \in (2r_3, 2)$, we find

$$\begin{aligned}
 f_{0,r_3}(x) &= \mathbb{P}(X \in dx \mid L_3 = 0, R_3 = r_3)/dx \\
 &= \int_{r_2} \mathbb{P}(R_2 \in dr_2, R_1 \in dx - r_2 \mid L_3 = 0, R_3 = r_3)/dx \\
 &= \frac{2}{(\ln r_3)^2} \int_{r_2=r_3 \vee (x-1)}^{x/2} (x - r_2)^{-1} r_2^{-1} dr_2 \\
 &= \frac{2}{\left(\ln \frac{1}{r_3}\right)^2} \frac{1}{x} \left[\ln \left(\frac{x - r_3}{r_3} \right) \mathbb{1}(2r_3 \leq x < 1 + r_3) \right. \\
 &\quad \left. + \ln \left(\frac{1}{x - 1} \right) \mathbb{1}(1 + r_3 \leq x < 2) \right]; \tag{4.17}
 \end{aligned}$$

we set $f_{0,r_3}(x) = 0$ for $x \notin (2r_3, 2)$.

Case 2: $l_3 > 0$ and $r_3 = 1$. This condition implies that $R_1 = R_2 = 1$. Invoking symmetry, we can skip the derivation and immediately write

$$\begin{aligned}
 f_{l_3,1}(x) &= \frac{2}{\left(\ln \frac{1}{1-l_3}\right)^2} \frac{1}{x} \left[\ln \left(\frac{x - 1 + l_3}{1 - l_3} \right) \mathbb{1}(2 - 2l_3 \leq x < 2 - l_3) \right. \\
 &\quad \left. + \ln \frac{1}{x - 1} \mathbb{1}(2 - l_3 \leq x < 2) \right] \tag{4.18}
 \end{aligned}$$

for $x \in (2 - 2l_3, 2)$; we set $f_{l_3,1}(x) = 0$ for $x \notin (2 - 2l_3, 2)$.

Case 3: $0 < l_3 < t < r_3 < 1$. There are six possible scenarios for the random vector $(L_1, R_1, L_2, R_2, L_3, R_3)$, and to help us discuss the cases, we consider values l_2, r_2 satisfying $0 < l_2 < l_3 < t < r_3 < r_2 < 1$.

(a) $L_1 = l_2, L_2 = L_3 = l_3$ and $R_1 = R_2 = 1, R_3 = r_3$.

In this subcase, we consider the event that the first pivot we choose locates between

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0 and l_3 , the second pivot has value l_3 , and the third pivot has value r_3 . Denote this event by E_{llr} (with llr indicating that we shrink the search intervals by moving the lefthand, lefthand, and then righthand endpoints). We have

$$\begin{aligned} \mathbb{P}(L_1 \in dl_2, R_1 = 1, L_2 \in dl_3, R_2 = 1, L_3 \in dl_3, R_3 \in dr_3) \\ = \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1) dl_2 \frac{dl_3}{1-l_2} \frac{dr_3}{1-l_3}. \end{aligned} \quad (4.19)$$

Integrating over all possible values of l_2 , we get

$$\begin{aligned} \mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{llr}) \\ = \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{1-l_3} \ln \left(\frac{1}{1-l_3} \right) dl_3 dr_3. \end{aligned}$$

(b) $L_1 = L_2 = 0, L_3 = l_3$ and $R_1 = r_2, R_2 = R_3 = r_3$.

In this and all subsequence subcases, we use notation like that in subcase (a). In this subcase, we invoke symmetry in comparison with subcase (a). The results are

$$\begin{aligned} \mathbb{P}(L_1 = 0, R_1 \in dr_2, L_2 = 0, L_3 \in dl_3, R_2 = R_3 \in dr_3) \\ = \mathbb{1}(0 < l_3 < t < r_3 < r_2 < 1) dr_2 \frac{dr_3}{r_2} \frac{dl_3}{r_3} \end{aligned} \quad (4.20)$$

and

$$\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{rrl}) = \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{r_3} \ln \left(\frac{1}{r_3} \right) dl_3 dr_3.$$

(c) $L_1 = L_2 = l_2, L_3 = l_3$ and $R_1 = 1, R_2 = R_3 = r_3$.

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In this subcase we have

$$\begin{aligned} & \mathbb{P}(R_1 = 1, L_1 = L_2 \in dl_2, L_3 \in dl_3, R_2 = R_3 \in dr_3) \\ &= \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1) dl_2 \frac{dr_3}{1-l_2} \frac{dl_3}{r_3-l_2}. \end{aligned} \quad (4.21)$$

Integrating over the possible values of l_2 , we find

$$\begin{aligned} & \mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{lrl}) / (dl_3 dr_3) \\ &= \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{1-r_3} \left[\ln \left(\frac{1}{r_3-l_3} \right) - \ln \left(\frac{1}{r_3} \right) - \ln \left(\frac{1}{1-l_3} \right) \right]. \end{aligned}$$

(d) $L_1 = 0, L_2 = L_3 = l_3$ and $R_1 = R_2 = r_2, R_3 = r_3$.

In this subcase, by symmetry with subcase (c), we have

$$\begin{aligned} & \mathbb{P}(L_1 = 0, L_2 = L_3 \in dl_3, R_1 = R_2 \in dr_2, R_3 \in dr_3) \\ &= \mathbb{1}(0 < l_3 < t < r_3 < r_2 < 1) dr_2 \frac{dl_3}{r_2} \frac{dr_3}{r_2-l_3}. \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} & \mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{rlr}) / (dl_3 dr_3) \\ &= \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{l_3} \left[\ln \left(\frac{1}{r_3-l_3} \right) - \ln \left(\frac{1}{1-l_3} \right) - \ln \left(\frac{1}{r_3} \right) \right]. \end{aligned}$$

(e) $L_1 = 0, L_2 = l_2, L_3 = l_3$ and $R_1 = R_2 = R_3 = r_3$.

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In this subcase we have

$$\begin{aligned} & \mathbb{P}(L_1 = 0, L_2 \in dl_2, L_3 \in dl_3, R_1 = R_2 = R_3 \in dr_3) \\ &= \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1) dr_3 \frac{dl_2}{r_3} \frac{dl_3}{r_3 - l_2}. \end{aligned} \quad (4.23)$$

Integrating over the possible values of l_2 , we have

$$\begin{aligned} & \mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{rll}) \\ &= \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{r_3} \left[\ln \left(\frac{1}{r_3 - l_3} \right) - \ln \left(\frac{1}{r_3} \right) \right] dl_3 dr_3. \end{aligned}$$

(f) $L_1 = L_2 = L_3 = l_3$ and $R_1 = 1, R_2 = r_2, R_3 = r_3$.

In this final subcase, by symmetry with subcase (e), we have

$$\begin{aligned} & \mathbb{P}(L_1 = L_2 = L_3 \in dl_3, R_1 = 1, R_2 \in dr_2, R_3 \in dr_3) \\ &= \mathbb{1}(0 < l_3 < t < r_3 < r_2 < 1) dl_3 \frac{dr_2}{1 - l_3} \frac{dr_3}{r_2 - l_3} \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} & \mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{lrr}) / (dl_3 dr_3) \\ &= \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{1 - l_3} \left[\ln \left(\frac{1}{r_3 - l_3} \right) - \ln \left(\frac{1}{1 - l_3} \right) \right]. \end{aligned}$$

Summing results from the six subcases, we conclude in Case 3 that

$$\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3) = \mathbb{1}(0 < l_3 < t < r_3 < 1) g(l_3, r_3) dl_3 dr_3, \quad (4.25)$$

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where

$$g(l_3, r_3) := \left[\frac{1}{l_3(1-l_3)} + \frac{1}{r_3(1-r_3)} \right] \ln \left(\frac{1}{r_3-l_3} \right) - \left(\frac{1}{l_3} + \frac{1}{1-r_3} \right) \left[\ln \left(\frac{1}{r_3} \right) + \ln \left(\frac{1}{1-l_3} \right) \right]. \quad (4.26)$$

The conditional joint distribution of (L_1, R_1, L_2, R_2) given $(L_3, R_3) = (l_3, r_3)$ can be derived by dividing each of (4.19)–(4.24) by (4.25), and we can then compute f_{l_3, r_3} from these conditional distributions. Let us write

$$f_{l_3, r_3}(x) = \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{g(l_3, r_3)} \sum_{i=1}^6 f_{l_3, r_3}^{(i)}(x), \quad (4.27)$$

where $f_{l_3, r_3}^{(i)}(x) dl_3 dr_3 dx$ is the contribution to

$$\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, X \in dx)$$

arising from the i th subcase of the six.

In subcase (a) we know that $X = R_1 - L_1 + R_2 - L_2 = 2 - l_2 - l_3$. Changing variables from l_2 to x , from (4.19) we find

$$f_{l_3, r_3}^{(1)}(x) = \mathbb{1}(2 - 2l_3 \leq x < 2 - l_3) \frac{1}{1 - l_3} \frac{1}{x - 1 + l_3}.$$

In subcase (b) we know that $X = r_2 + r_3$. Changing variables from r_2 to x , from (4.20) we find

$$f_{l_3, r_3}^{(2)}(x) = \mathbb{1}(2r_3 \leq x < 1 + r_3) \frac{1}{r_3} \frac{1}{x - r_3}.$$

In subcase (c), we know that $X = 1 - 2l_2 + r_3$. Changing variables from l_2 to x ,

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from (4.21) we find

$$f_{l_3, r_3}^{(3)}(x) = \mathbb{1}(1 + r_3 - 2l_3 \leq x < 1 + r_3) \frac{1}{x + 1 - r_3} \frac{2}{x + r_3 - 1}.$$

In subcase (d), we know that $X = 2r_2 - l_3$. Changing variables from r_2 to x , from (4.22) we find

$$f_{l_3, r_3}^{(4)}(x) = \mathbb{1}(2r_3 - l_3 \leq x < 2 - l_3) \frac{2}{x + l_3} \frac{1}{x - l_3}.$$

In subcase (e), we know that $X = 2r_3 - l_2$. Changing variables from l_2 to x , from (4.23) we find

$$f_{l_3, r_3}^{(5)}(x) = \mathbb{1}(2r_3 - l_3 \leq x < 2r_3) \frac{1}{r_3} \frac{1}{x - r_3}.$$

Finally, in subcase (f), we know that $X = 1 + r_2 - l_3$. Changing variables from r_2 to x , from (4.24) we find

$$f_{l_3, r_3}^{(6)}(x) = \mathbb{1}(r_3 - 2l_3 + 1 \leq x < 2 - 2l_3) \frac{1}{1 - l_3} \frac{1}{x - 1 + l_3}.$$

The density functions f_{0, r_3} and $f_{l_3, 1}$ we have found in Cases 1 and 2 are continuous. We have chosen to make the functions $f_{l_3, r_3}^{(i)}$ (for $i = 1, \dots, 6$) right continuous in Case 3. Thus the density f_{l_3, r_3} we have determined at (4.27) in Case 3 is right continuous. \square

Our next lemma handles the cases $t = 0$ and $t = 1$ that were excluded from Lemma 4.3, and its proof is the same as for Cases 1 and 2 in the proof of Lemma 4.3.

Lemma 4.4. (a) *Suppose $t = 0$. Let $0 < r_3 < 1$. Conditionally given $(L_3, R_3) =$*

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$(0, r_3)$, the random variable $X = \Delta_1 + \Delta_2$ has the right continuous density f_{0,r_3} specified in the sentence containing (4.17).

(b) Suppose $t = 1$. Let $0 < l_3 < 1$. Conditionally given $(L_3, R_3) = (l_3, 1)$, the random variable $X = \Delta_1 + \Delta_2$ has the right continuous density $f_{l_3,1}$ specified in the sentence containing (4.18).

We need to check the trivariate measurability of the function $f_t(l_3, r_3, x) := f_{l_3, r_3}(x)$ before diving into the derivation of the density function of J . Given a topological space S , let $\mathcal{B}(S)$ denote its Borel σ -field, that is, the σ -field generated by the open sets of S . Also, given $0 < t < 1$, let

$$S_t := \{(l_3, r_3) \neq (0, 1) : 0 \leq l_3 < t < r_3 \leq 1\}.$$

Lemma 4.5. (a) For $0 < t < 1$, the conditional density function $f_t(l_3, r_3, x)$, formed to be a right continuous function of x , is measurable with respect to $\mathcal{B}(S_t \times \mathbb{R})$.

(b) For $t = 0$, the conditional density function $f_0(r_3, x) := f_{0,r_3}(x)$, continuous in x , is measurable $\mathcal{B}((0, 1) \times \mathbb{R})$.

(c) For $t = 1$, the conditional density function $f_1(l_3, x) := f_{l_3,1}(x)$, continuous in x , is measurable $\mathcal{B}((0, 1) \times \mathbb{R})$.

We introduce (the special case of real-valued f of) a lemma taken from Gowrisankaran [19, Theorem 3] giving a sufficient condition for the measurability of certain functions f defined on product spaces. The lemma will help us prove Lemma 4.5.

Lemma 4.6 (Gowrisankaran [19]). Let (Ω, \mathcal{F}) be a measurable space. Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$. Suppose that the section mapping $f(\cdot, y)$ is \mathcal{F} -measurable for each $y \in \mathbb{R}$ and that the section mapping $f(\omega, \cdot)$ is either right continuous for each $\omega \in \Omega$ or left continuous for each $\omega \in \Omega$. Then f is measurable with respect to the product σ -field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

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Proof of Lemma 4.5. We prove (b), then (c), and finally (a).

(b) Recall the expression (4.17) for $f_0(r_3, x)$ [for $0 < r_3 < 1$ and $x \in (2r_3, 2]$]. We apply Lemma 4.6 with $(\Omega, \mathcal{F}) = ((0, 1), \mathcal{B}((0, 1)))$. The right continuity of $f_0(r_3, \cdot)$ has already been established in Lemma 4.4(a). On the other hand, when we fix x and treat $f(0, r_3, x)$ as a function of r_3 , the conditional density function can be separated into the following cases:

- If $x \leq 0$ or $x \geq 2$, then $f_0(r_3, x) \equiv 0$.
- If $0 < x < 2$, then from (4.17) we see that $f_0(r_3, x)$ is piecewise continuous (with a finite number of measurable domain-intervals), and hence measurable, in r_3 .

Since the product σ -field $\mathcal{B}((0, 1)) \otimes \mathcal{B}(\mathbb{R})$ equals $\mathcal{B}((0, 1) \times \mathbb{R})$, the desired result follows.

(c) Assertion (c) can be proved by a similar argument or by invoking symmetry.

(a) We apply Lemma 4.6 with $(\Omega, \mathcal{F}) = (S_t, \mathcal{B}(S_t))$. The right continuity of $f(l_3, r_3, \cdot)$ has already been established in Lemma 4.3, so it suffices to show for each $x \in \mathbb{R}$ that $f(l_3, r_3, x)$ is measurable in $(l_3, r_3) \in S_t$. For this it is clearly sufficient to show that $f(0, r_3, x)$ is measurable in $r_3 \in (t, 1)$, that $f(l_3, 1, x)$ is measurable in $l_3 \in (0, t)$, and that $f(l_3, r_3, x)$ is measurable in $(l_3, r_3) \in (0, t) \times (t, 1)$. All three of these assertions follow from the fact that piecewise continuous functions (with a finite number of measurable domain-pieces) are measurable; in particular, for the third assertion, note that the function g defined at (4.26) is continuous in $(l_3, r_3) \in (0, t) \times (t, 1)$ and that each of the six expressions $f_{l_3, r_3}^{(i)}(x)$ appearing in (4.27) is piecewise continuous (with a finite number of measurable domain-pieces) in these values of (l_3, r_3) for each fixed $x \in \mathbb{R}$.

This complete the proof. □

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As explained at the outset of this section, a conditional density $h_{l_3, r_3}(\cdot)$ for $J(t)$ given $(L_3, R_3) = (l_3, r_3)$ can now be formed by convolving the conditional density function of X , namely $f_{l_3, r_3}(\cdot)$, with the conditional distribution function of Y . That is, we can write

$$h_{l_3, r_3}(x) = \int f_{l_3, r_3}(x - y) \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l_3, r_3)). \quad (4.28)$$

We now prove in the next two lemmas that the joint measurability of $f_{l_3, r_3}(x)$ with respect to (l_3, r_3, x) ensures the same for $h_{l_3, r_3}(x)$.

Lemma 4.7. *Let (Ω, \mathcal{F}) be a measurable space. Let $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function measurable with respect to the product σ -field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$. Let V and Y be two measurable functions defined on a common measurable space and taking values in Ω and \mathbb{R} , respectively. Then a conditional probability distribution $\mathbb{P}(Y \in dy \mid V)$ for Y given V exists, and the function $Tg : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$Tg(v, x) := \int g(v, x - y) \mathbb{P}(Y \in dy \mid V = v)$$

is measurable with respect to the product σ -field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

Proof. Since Y is a real-valued random variable, by Billingsley [3, Theorem 33.3] or Durrett [8, Theorem 4.1.18] there exists a conditional probability distribution for Y given V . Consider the restricted collection

$$\mathcal{H} := \{f \geq 0 : Tf \text{ is measurable } \mathcal{F} \otimes \mathcal{B}(\mathbb{R})\}$$

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of functions defined on $\Omega \times \mathbb{R}$ and the π -system

$$\mathcal{A} := \{B_1 \times B_2 : B_1 \in \mathcal{F} \text{ and } B_2 \in \mathcal{B}(\mathbb{R})\}$$

of all measurable rectangles in $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$. If we show that the indicator function $\mathbb{1}(A)$ is in \mathcal{H} for every $A \in \mathcal{A}$, it then follows from the monotone convergence theorem and the monotone class theorem in Durrett [8, Theorem 5.2.2] that \mathcal{H} contains all nonnegative functions measurable with respect to $\sigma(\mathcal{A}) = \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$, as desired.

We thus let $A = B_1 \times B_2 \in \mathcal{A}$ for some $B_1 \in \mathcal{F}$ and $B_2 \in \mathcal{B}(\mathbb{R})$. Then

$$T\mathbb{1}(A)(v, x) = \mathbb{1}_{B_1}(v) \mathbb{P}(Y \in x - B_2 \mid V = v).$$

We claim that

$$(v, x) \mapsto \mathbb{P}(Y \in x - B \mid V = v) \text{ is } \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \text{ measurable,} \quad (4.29)$$

for any $B \in \mathcal{B}(\mathbb{R})$ and thus $\mathbb{1}(A) \in \mathcal{H}$; so the proof of the claim (4.29) will complete the proof of the lemma.

Since the collection of sets $B \in \mathcal{B}(\mathbb{R})$ satifying (4.29) is a λ -system, we need only check (4.29) for intervals of the form $B = [a, b)$ with $b > a$; we can then apply Dynkin's π - λ theorem to complete the proof of the claim. For fixed x , by Billingsley [3, Theorem 34.5] the mapping $v \mapsto \mathbb{P}(Y \in x - B \mid V = v)$ is a version of $\mathbb{E}[\mathbb{1}_B(x - Y) \mid V]$ and so is \mathcal{F} -measurable. Furthermore, for fixed $v \in \Omega$ and $x, z \in \mathbb{R}$

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with $z > x$ and z close enough to x , we have

$$\begin{aligned} & |\mathbb{P}(Y \in x - B \mid V = v) - \mathbb{P}(Y \in z - B \mid V = v)| \\ & \leq \mathbb{P}(Y \in (x - b, z - b] \mid V = v) + \mathbb{P}(Y \in (x - a, z - a] \mid V = v), \end{aligned}$$

and the bound here is small by the right continuity of the conditional distribution function. We have established right continuity of $\mathbb{P}(Y \in x - B \mid V = v)$ in x for fixed v , and thus we can apply Lemma 4.6 to conclude that $(v, x) \mapsto \mathbb{P}(Y \in x - B \mid V = v)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$ measurable. This completes the proof of the lemma. \square

We can now handle the measurability of $(l_3, r_3, x) \mapsto h_{l_3, r_3}(x)$.

Lemma 4.8.

- (a) For $0 < t < 1$, the mapping $(l_3, r_3, x) \mapsto h_{l_3, r_3}(x)$ is $\mathcal{B}(S_t \times \mathbb{R})$ measurable.
- (b) For $t = 0$, the mapping $(r_3, x) \mapsto h_{0, r_3}(x)$ is $\mathcal{B}((0, 1) \times \mathbb{R})$ measurable.
- (c) For $t = 1$, the mapping $(l_3, x) \mapsto h_{l_3, 1}(x)$ is $\mathcal{B}((0, 1) \times \mathbb{R})$ measurable.

Proof. We prove (a); the claims (b) and (c) are proved similarly. Choosing $\Omega = S_t$ and $g(l_3, r_3, x) = f_{l_3, r_3}(x)$ with $(l_3, r_3) \in \Omega$ in Lemma 4.7, we conclude that

$$(l_3, r_3, x) \mapsto h_{l_3, r_3}(x) = Tg(l_3, r_3, x)$$

is $\mathcal{B}(S_t \times \mathbb{R})$ measurable. \square

Recall the definition of $f_t(x)$ at (4.13). It then follows from Lemma 4.8 that $f_t(x)$ is well defined and measurable with respect to $x \in \mathbb{R}$ for fixed $0 \leq t \leq 1$. This completes the proof of Theorem 4.2.

Chapter 5

Boundedness and continuity of f_t

5.1 Uniform boundedness of the density functions

In this section, we prove that the functions f_t are uniformly bounded for $0 \leq t \leq 1$.

Theorem 5.1. *The densities f_t are uniformly bounded by 10 for $0 \leq t \leq 1$.*

The proof of Theorem 5.1 is our later Lemmas 5.2 and 5.6. In particular, the numerical value 10 comes from the bound in the last line of the proof of Lemma 5.2 plus two times the bound in the last sentence of the proof of Lemma 5.6. A bound on the function f_t is established by first finding a bound on the conditional density function f_{l_3, r_3} . Observe that the expressions in the proof of Lemma 4.3 for $f_{l_3, r_3}^{(i)}(x)$ (for $i = 1, \dots, 6$) in Case 3 all involve indicators of intervals. The six endpoints of these intervals are

$$2r_3 - l_3, 2r_3, 1 + r_3 - 2l_3, 1 + r_3, 2 - 2l_3, \text{ and } 2 - l_3,$$

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with $0 < l_3 < t < r_3 < 1$. The relative order of these six endpoints is determined once we know the value of $\rho = \rho(l_3, r_3) := l_3/(1 - r_3)$. Indeed:

- When $\rho \in (0, 1/2)$, the order is

$$2r_3 - l_3 < 2r_3 < 1 + r_3 - 2l_3 < 1 + r_3 < 2 - 2l_3 < 2 - l_3.$$

- When $\rho \in (1/2, 1)$, the order is

$$2r_3 - l_3 < 1 + r_3 - 2l_3 < 2r_3 < 2 - 2l_3 < 1 + r_3 < 2 - l_3.$$

- When $\rho \in (1, 2)$, the order is

$$1 + r_3 - 2l_3 < 2r_3 - l_3 < 2 - 2l_3 < 2r_3 < 2 - l_3 < 1 + r_3.$$

- When $\rho \in (2, \infty)$, the order is

$$1 + r_3 - 2l_3 < 2r_3 - l_3 < 2 - 2l_3 < 2 - l_3 < 2r_3 < 1 + r_3.$$

When $\rho = 0$ (i.e., in Case 1 in the proof of Lemma 4.3: $l_3 = 0 < r_3 < 1$), the function f_{l_3, r_3} is given by f_{0, r_3} at (4.17). When $\rho = \infty$ (i.e., in Case 2 in the proof of Lemma 4.3: $0 < l_3 < r_3 = 1$), the function f_{l_3, r_3} is given by $f_{l_3, 1}$ at (4.18). The result (4.27) for Case 3 in the proof of Lemma 4.3 can be reorganized as follows,

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where we define the following functions to simplify notation:

$$\begin{aligned}
 m_1(x, l_3, r_3) &:= \frac{1}{r_3(x - r_3)} + \frac{2}{(x + l_3)(x - l_3)}, \\
 m_2(x, l_3, r_3) &:= \frac{1}{(1 - l_3)(x - 1 + l_3)} + \frac{2}{(x + l_3)(x - l_3)}, \\
 m_3(x, l_3, r_3) &:= \frac{2}{(x + 1 - r_3)(x + r_3 - 1)} + \frac{1}{(1 - l_3)(x + l_3 - 1)}, \\
 m_4(x, l_3, r_3) &:= \frac{1}{r_3(x - r_3)} + \frac{2}{(x + 1 - r_3)(x + r_3 - 1)}.
 \end{aligned}$$

When $\rho \in (0, 1)$, the conditional density f_{l_3, r_3} satisfies

$$\begin{aligned}
 f_{l_3, r_3}(x) g(l_3, r_3) &= \mathbb{1}(2r_3 - l_3 \leq x < 1 + r_3 - 2l_3) m_1(x, l_3, r_3) \\
 &\quad + \mathbb{1}(1 + r_3 - 2l_3 \leq x < 1 + r_3) [m_2(x, l_3, r_3) + m_4(x, l_3, r_3)] \\
 &\quad + \mathbb{1}(1 + r_3 \leq x < 2 - l_3) m_2(x, l_3, r_3).
 \end{aligned} \tag{5.1}$$

Lastly, when $\rho \in (1, \infty)$, the conditional density f_{l_3, r_3} satisfies

$$\begin{aligned}
 f_{l_3, r_3}(x) g(l_3, r_3) &= \mathbb{1}(1 + r_3 - 2l_3 \leq x < 2r_3 - l_3) m_3(x, l_3, r_3) \\
 &\quad + \mathbb{1}(2r_3 - l_3 \leq x < 2 - l_3) [m_2(x, l_3, r_3) + m_4(x, l_3, r_3)] \\
 &\quad + \mathbb{1}(2 - l_3 \leq x < 1 + r_3) m_4(x, l_3, r_3).
 \end{aligned} \tag{5.2}$$

Recall the definition of $f_t(x)$ at (4.13). For any $x \in \mathbb{R}$ we can decompose $f_t(x)$

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into three contributions:

$$\begin{aligned} f_t(x) &= \mathbb{E} h_{L_3, R_3}(x) \\ &= \mathbb{E}[h_{L_3, R_3}(x); \rho(L_3, R_3) = 0] + \mathbb{E}[h_{L_3, R_3}(x); \rho(L_3, R_3) = \infty] \\ &\quad + \mathbb{E}[h_{L_3, R_3}(x); 0 < \rho(L_3, R_3) < \infty]. \end{aligned}$$

We first consider the contribution from the case $0 < \rho(L_3, R_3) < \infty$ for any $0 < t < 1$, noting that this case doesn't contribute to $f_t(x)$ when $t = 0$ or $t = 1$.

Define

$$b(l_3, r_3) = \frac{1}{g(l_3, r_3)} \frac{3}{2} \left[\frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)} \right]. \quad (5.3)$$

Lemma 5.2. *The contribution to the density function f_t from the case $0 < \rho(L_3, R_3) < \infty$ is uniformly bounded for $0 < t < 1$. More precisely, given $0 < t < 1$ and $0 < l_3 < t < r_3 < 1$, we have*

$$f_{l_3, r_3}(x) \leq b(l_3, r_3) \quad \text{and} \quad h_{l_3, r_3}(x) \leq b(l_3, r_3) \quad \text{for all } x \in \mathbb{R}; \quad (5.4)$$

and, moreover, $\mathbb{E}[h_{L_3, R_3}(x); 0 < \rho(L_3, R_3) < \infty]$ is uniformly bounded for $0 < t < 1$.

Proof. For (5.4), we need only establish the bound on f .

We start to bound (5.1) for $0 < \rho < 1$. The function m_1 is a decreasing function of x and thus reaches its maximum in (5.1) when $x = 2r_3 - l_3$:

$$m_1(x, l_3, r_3) \leq m_1(2r_3 - l_3, l_3, r_3) = \frac{3}{2} \frac{1}{r_3(r_3 - l_3)}.$$

The function $m_2 + m_4$ is also a decreasing function of x , and the maximum in (5.1) occurs at $x = 1 + r_3 - 2l_3$. Plug in this x -value and use the fact that $1 - l_3 > r_3$ when

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$\rho < 1$ to obtain

$$(m_2 + m_4)(x, l_3, r_3) \leq \frac{3}{2} \frac{1}{r_3(r_3 - l_3)} + \frac{3}{2} \frac{1}{(1 - l_3)(r_3 - l_3)}.$$

Finally, the function m_2 is again a decreasing function of x , and the maximum in (5.1) occurs at $x = 1 + r_3$. Plug in this x -value and use the facts that $1 + r_3 - l_3 > 2r_3$ and $1 + r_3 + l_3 > r_3 - l_3$ to conclude

$$m_2(x, l_3, r_3) \leq \frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)}.$$

By the above three inequalities, we summarize that for $0 < \rho < 1$ we have for all x the inequality

$$f_{l_3, r_3}(x) \leq \frac{1}{g(l_3, r_3)} \frac{3}{2} \left[\frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)} \right].$$

The method to upper-bound (5.2) is similar to that for (5.1), or one can again invoke symmetry, and we skip the proof here.

For the expectation of $b(L_3, R_3)$, we see immediately that

$$\begin{aligned} \mathbb{E}[b(L_3, R_3); 0 < \rho(L_3, R_3) < \infty] &= \frac{3}{2} \int_0^t \int_t^1 \left[\frac{1}{r(r-l)} + \frac{1}{(1-l)(r-l)} \right] dr dl \\ &= \frac{\pi^2}{4} + \frac{3}{2} (\ln t) [\ln(1-t)] \leq \frac{\pi^2}{4} + \frac{3}{2} (\ln 2)^2. \end{aligned}$$

□

For the cases $\rho(L_3, R_3) = 0$ and $\rho(L_3, R_3) = \infty$, we cannot find a constant bound $b(l_3, r_3)$ on the function f_{l_3, r_3} such that the corresponding contributions to $\mathbb{E} b(L_3, R_3)$ are bounded for $0 \leq t \leq 1$. Indeed, although we shall omit the proof since it would

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take us too far afield, there exists no such bound $b(l_3, r_3)$.

Instead, to prove the uniform boundedness of the contributions in these two cases, we take a different approach. The following easily-proved lemma comes from Grübel and Rösler [21, proof of Theorem 9].

Lemma 5.3. *Consider a sequence of independent random variables V_1, V_2, \dots , each uniformly distributed on $(1/2, 1)$, and let*

$$V := 1 + \sum_{n=1}^{\infty} \prod_{k=1}^n V_k. \quad (5.5)$$

Then the random variables $Z(t)$, $0 \leq t \leq 1$, defined at (4.10) are all stochastically dominated by V . Furthermore, $\mathbb{E}V = 4$; and V has everywhere finite moment generating function m and therefore superexponential decay in the right tail, in the sense that for any $\theta \in (0, \infty)$ we have $\mathbb{P}(V \geq x) = o(e^{-\theta x})$ as $x \rightarrow \infty$.

The following lemma pairs the stochastic upper bound V on $Z(t)$ with a stochastic lower bound. These stochastic bounds will be useful in later sections.

Lemma 5.4. *Let D be a random variable following the Dickman distribution with support $[1, \infty)$. Then for all $0 \leq t \leq 1$ we have $D \leq Z(t) \leq V$ stochastically.*

Proof. Recall that $\Delta_1(t) = R_1(t) - L_1(t)$. We first use a coupling argument to show that $\Delta_1(t)$ is stochastically increasing for $0 \leq t \leq 1/2$. Let $U = U_1 \sim \text{uniform}(0, 1)$ be the first key in the construction of Z as described in (4.7)–(4.10). Let $0 \leq t_1 < t_2 \leq 1/2$. It is easy to see that $\Delta_1(t_1) = \Delta_1(t_2)$ unless $t_1 < U < t_2$, in which case $\Delta_1(t_1) = U < t_2 \leq 1/2 \leq 1 - t_2 < 1 - U = \Delta_1(t_2)$.

Let $V_1 \sim \text{uniform}(1/2, 1)$, as in Lemma 5.3, and let $0 \leq t \leq 1/2$. Since $\Delta_1(0) \stackrel{\mathcal{L}}{=} U$ and $\Delta_1(1/2) \stackrel{\mathcal{L}}{=} V_1$, we immediately have $U \leq \Delta_1(t) \leq V_1$ stochastically. This implies

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by a simple induction argument on k involving the conditional distribution of $\Delta_k(t)$ given $\Delta_{k-1}(t)$ that $D \leq Z(t) \leq V$ stochastically. \square

Remark 5.5. (a) Note that we do *not* claim that $Z(t)$ is stochastically increasing in $t \in [0, 1/2]$. Indeed, other than the stochastic ordering $D = Z(0) \leq Z(t)$, we do not know whether any stochastic ordering relations hold among the random variables $Z(t)$.

(b) The random variable V can be interpreted as a sort of “limiting greedy (or ‘on-line’) worst-case **QuickQuant** normalized key-comparisons cost”. Indeed, if upon each random bisection of the search interval one always chooses the half of greater length and sums the lengths to get $V^{(n)}$, then the limiting distribution of $V^{(n)}/n$ is that of V .

Lemma 5.6. *The contributions to the density function f_t from the cases $\rho(L_3, R_3) = 0$ and $\rho(L_3, R_3) = \infty$ are uniformly bounded for $0 \leq t \leq 1$.*

Proof. Because the Dickman density is bounded above by $e^{-\gamma}$, we need only consider $0 < t < 1$. The case $\rho(L_3(t), R_3(t)) = 0$ corresponds to $L_3(t) = 0$, while the case $\rho(L_3(t), R_3(t)) = \infty$ corresponds to $R_3(t) = 1$. By symmetry, the contribution from $R_3(t) = 1$ is the same as the contribution from $L_3(1 - t) = 0$, so we need only show that the contribution from $L_3(t) = 0$ is uniformly bounded. We will do this by showing that the larger contribution from $L_2(t) = 0$ is uniformly bounded.

By conditioning on the value of $R_2(t)$, the contribution from $L_2(t) = 0$ is

$$\begin{aligned} c_t(x) &:= \mathbb{P}(L_2(t) = 0, J(t) \in dx) / dx \\ &= \int_{r \in (t, 1)} \int_{z > 1} \mathbf{1}(r \leq x - rz < 1) (x - rz)^{-1} \mathbb{P}(Z(t/r) \in dz) dr. \end{aligned} \quad (5.6)$$

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If $z > 1$ and $r \leq x - rz$, then $(x/r) - 1 \geq z > 1$ and so $r < x/2$. Therefore we find

$$c_t(x) \leq \int_t^{\min\{1, x/2\}} \int_{\max\{1, (x-1)/r\} < z \leq (x/r)-1} (x - rz)^{-1} \mathbb{P}(Z(t/r) \in dz) dr.$$

The integrand (including the implicit indicator function) in the inner integral is an increasing function of z over the interval $(-\infty, (x/r) - 1]$, with value r^{-1} at the upper endpoint of the interval. We can thus extend it to a nondecreasing function $\phi \equiv \phi_{x,r}$ with domain \mathbb{R} by setting $\phi(z) = r^{-1}$ for $z > (x/r) - 1$. It then follows that

$$\begin{aligned} c_t(x) &\leq \int_0^{\min\{1, x/2\}} \left[\int_{\max\{1, (x-1)/r\} < z \leq (x/r)-1} (x - rz)^{-1} \mathbb{P}(V \in dz) \right. \\ &\quad \left. + r^{-1} \mathbb{P}\left(V > \frac{x}{r} - 1\right) \right] dr \\ &\leq \int_0^{\min\{1, x/2\}} \int_{\max\{1, (x-1)/r\} < z \leq (x/r)-1} (x - rz)^{-1} \mathbb{P}(V \in dz) dr \\ &\quad + \int_0^{x/2} r^{-1} \mathbb{P}\left(V > \frac{x}{r} - 1\right) dr. \end{aligned} \tag{5.7}$$

By the change of variables $v = (x/r) - 1$, the second term in (5.7) equals

$$\int_1^\infty (v + 1)^{-1} \mathbb{P}(V > v) dv \leq \frac{1}{2} \int_0^\infty \mathbb{P}(V > v) dv = \frac{1}{2} \mathbb{E} V = 2.$$

Comparing the integrals $c_t(x)$ at (5.6) and the first term in (5.7), we see that the only constraint that has been discarded is $r > t$. We therefore see by the same argument that produces (5.6) that the first term in (5.7) is the value of the density for $W := U_1(1 + U_2V)$ at x , where U_1 , U_2 , and V are independent and U_1 and U_2 are uniformly distributed on $(0, 1)$. Thus to obtain the desired uniform boundedness of f_t we need only show that W has a bounded density. For that, it suffices to observe that the conditional density of W given U_2 and V is bounded above by 1 (for any

values of U_2 and V), and so the unconditional density is bounded by 1. We conclude that $c_t(x) \leq 3$, and this completes the proof. \square

Remark 5.7. Based on simulation results, we conjecture that the density functions f_t are uniformly bounded by $e^{-\gamma}$ (the sup-norm of the continuous Dickman density f_0) for $0 \leq t \leq 1$.

5.2 Uniform continuity of the density function f_t

From the previous section, we know that for $0 < t < 1$ in the case $0 < l < t < r < 1$ (i.e., the case $0 < \rho < \infty$) the function $f_{l,r}$ is càdlàg (that is, a right continuous function with left limits) and bounded above by $b(l, r)$, where the corresponding contribution $\mathbb{E}[b(L_3, R_3); 0 < \rho(L_3, R_3) < \infty]$ is finite. Applying the dominated convergence theorem, we conclude that the contribution to f_t from this case is also càdlàg.

For the cases $0 = l < t < r < 1$ ($\rho = 0$) and $0 < l < t < r = 1$ ($\rho = \infty$), the functions $f_{0,r}$ and $f_{l,1}$ are both continuous on the real line. In this section, we will build bounds $b_t(l, r)$ (note that these bounds depend on t) for these two cases (Lemma 5.8 for $\rho = 0$ and Lemma 5.9 for $\rho = \infty$) in similar fashion as for Lemma 5.6 such that both $\mathbb{E}[b_t(L_3, R_3); \rho(L_3, R_3) = 0]$ and $\mathbb{E}[b_t(L_3, R_3); \rho(L_3, R_3) = \infty]$ are finite for any $0 < t < 1$. Given these bounds, we can apply the dominated convergence theorem to conclude that the density f_t is càdlàg. Later, this result will be sharpened substantially in Theorem 5.11.

Let $\alpha \approx 3.59112$ be the unique real solution of $1 + x - x \ln x = 0$ and let $\beta :=$

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$1/\alpha \approx 0.27846$. Define

$$b_1(r) := \frac{2}{\ln r^{-1}} \frac{1}{1+r} \quad \text{and} \quad b_2(r) := \frac{2}{(\ln r^{-1})^2} \frac{1}{r} \beta.$$

Lemma 5.8. *Suppose $\rho = 0$, i.e., $0 = l_3 < t < r_3 < 1$. If $t \geq \beta$, then the optimal constant upper bound on f_{l_3, r_3} is*

$$b_t(l_3, r_3) = b_1(r_3),$$

with corresponding contribution

$$\mathbb{E}[b_t(L_3(t), R_3(t)); \rho(L_3(t), R_3(t)) = 0] = \int_t^1 \frac{\ln r^{-1}}{1+r} dr \leq \int_\beta^1 \frac{\ln r^{-1}}{1+r} dr < \frac{1}{4}$$

to $\mathbb{E} b_t(L_3(t), R_3(t))$. If $t < \beta$, then the optimal constant upper bound on f_{l_3, r_3} is the continuous function

$$b_t(l_3, r_3) = b_1(r_3) \mathbf{1}(\beta \leq r_3 < 1) + b_2(r_3) \mathbf{1}(t < r_3 < \beta)$$

of $r_3 \in (t, 1]$, with corresponding contribution

$$\begin{aligned} \mathbb{E}[b_t(L_3(t), R_3(t)); \rho(L_3(t), R_3(t)) = 0] &= \int_\beta^1 \frac{\ln r^{-1}}{1+r} dr + \beta(\ln \beta - \ln t) \\ &< \frac{1}{4} + \beta(\ln \beta - \ln t). \end{aligned}$$

Lemma 5.9. *Suppose $\rho = \infty$, i.e., $0 < l_3 < t < r_3 = 1$. If $t \leq 1 - \beta$, then the optimal constant upper bound on f_{l_3, r_3} is*

$$b_t(l_3, r_3) = b_1(1 - l_3),$$

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with corresponding contribution

$$\mathbb{E}[b_t(L_3(t), R_3(t)); \rho(L_3(t), R_3(t)) = 0] = \int_{1-t}^1 \frac{\ln r^{-1}}{1+r} dr \leq \int_{\beta}^1 \frac{\ln r^{-1}}{1+r} dr < \frac{1}{4}.$$

If $t > 1 - \beta$, then the optimal constant upper bound on f_{l_3, r_3} is the continuous function

$$b_t(l_3, r_3) = b_1(1 - l_3)\mathbf{1}(0 < l_3 \leq 1 - \beta) + b_2(1 - l_3)\mathbf{1}(1 - \beta < l_3 < t)$$

of $l_3 \in [0, t)$, with corresponding contribution

$$\begin{aligned} \mathbb{E}[b_t(L_3(t), R_3(t)); \rho(L_3(t), R_3(t)) = 0] &= \int_{\beta}^1 \frac{\ln r^{-1}}{1+r} dr + \beta[\ln \beta - \ln(1 - r)] \\ &< \frac{1}{4} + \beta[\ln \beta - \ln(1 - t)]. \end{aligned}$$

We prove Lemma 5.8 here, and Lemma 5.9 follows similarly or by symmetry.

Proof of Lemma 5.8. When $\rho = 0$, we have $l_3 = 0$, and the conditional density function is f_{0, r_3} in (4.17). The expression in square brackets at (4.17) is continuous and unimodal in x , with maximum value at $x = 1 + r_3$. Because the factor $1/x$ is decreasing, it follows that the maximum value of $f_{0, r_3}(x)$ is the maximum of

$$\frac{2}{(\ln r_3^{-1})^2} \frac{1}{x} \ln \left(\frac{x - r_3}{r_3} \right)$$

over $x \in [2r_3, 1 + r_3]$, i.e., the maximum of

$$\frac{2}{r_3 (\ln r_3^{-1})^2} \frac{\ln y}{1 + y}$$

over $y \in [1, 1/r_3]$. A simple calculation shows that the displayed expression is strictly

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increasing for $y \in [1, \alpha]$ and strictly decreasing for $y \in [\alpha, \infty)$. Thus the maximum for $y \in [1, 1/r_3]$ is achieved at $y = \alpha$ if $\alpha \leq 1/r_3$ and at $y = 1/r_3$ if $\alpha \geq 1/r_3$. Equivalently, $f_{0,r_3}(x)$ is maximized at $x = r_3(\alpha + 1)$ if $r_3 \leq \beta$ and at $x = 1 + r_3$ if $r_3 \geq \beta$. The claims about the optimal constant upper bound on f_{l_3,r_3} and the contribution to $\mathbb{E} b_t(L_3(t), R_3(t))$ now follow readily. \square

Remark 5.10. If we are not concerned about finding the best possible upper bound, then for the case $\rho = 0$ we can choose $b_t(l, r) := b_2(r)$; for the case $\rho = \infty$, we can choose $b_t(l, r) := b_2(1 - l)$. These two bounds still get us the desired finiteness of the contributions to $\mathbb{E} b_t(L_3, R_3)$ for any $0 < t < 1$.

Theorem 5.11. *For $0 < t < 1$, the density function $f_t : \mathbb{R} \rightarrow [0, \infty)$ is uniformly continuous.*

Proof. Fix $0 < t < 1$. By the dominated convergence theorem, the contributions to $f_t(x)$ from $0 = l < t < r < 1$ and $0 < l < t < r = 1$, namely, the functions

$$c_0(x) := \int_{r,y} f_{0,r}(x - y) \mathbb{P}(L_3(t) = 0, R_3(t) \in dr, Y \in dy)$$

and

$$c_1(x) := \int_{l,y} f_{l,1}(x - y) \mathbb{P}(L_3(t) \in dl, R_3(t) = 1, Y \in dy),$$

are continuous for $x \in \mathbb{R}$. Further, according to (4.25) and (4.27), the contribution from $0 < l < t < r < 1$ is $\sum_{i=1}^6 c^{(i)}(x)$, where we define

$$c^{(i)}(x) := \int_{l,r,y} f_{l,r}^{(i)}(x - y) \mathbb{P}(Y \in dy \mid (L_3(t), R_3(t)) = (l, r)) dl dr.$$

It is easy to see that all the functions c_0 , c_1 , and $c^{(i)}$ for $i = 1, \dots, 6$ vanish for arguments $x \leq 0$. To prove the uniform continuity of $f_t(x)$ for $x \in \mathbb{R}$, it thus suffices

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to show that each of the six functions $c^{(i)}$ for $i = 1, \dots, 6$ is continuous on the real line and that each of the eight functions c_0 , c_1 , and $c^{(i)}$ for $i = 1, \dots, 6$ vanishes in the limit as argument $x \rightarrow \infty$.

Fix $i \in \{1, \dots, 6\}$. Continuity of $c^{(i)}$ holds since $f_{l,r}^{(i)}$ is bounded by $b(l, r)$ defined at (5.3) and is continuous except at the boundary of its support. To illustrate, consider, for example, $i = 3$. For each fixed $0 < l < t < r < 1$ and $x \in \mathbb{R}$, we have $f_{l,r}^{(3)}(x + h - y) \rightarrow f_{l,r}^{(3)}(x - y)$ as $h \rightarrow 0$ for all but two exceptional values of y , namely, $y = x - (1 + r - 2l)$ and $y = x - (1 - r)$. From the discussion following (4.12) and from Theorem 4.2, we know that the conditional law of Y given $(L_3(t), R_3(t)) = (l, r)$ has a density with respect to Lebesgue measure, and hence the set of two exceptional points has zero measure under this law. We conclude from the dominated convergence theorem that

$$\int_y f_{l,r}^{(i)}(x - y) \mathbb{P}(Y \in dy \mid (L_3(t), R_3(t)) = (l, r))$$

is continuous in $x \in \mathbb{R}$. It now follows by another application of the dominated convergence theorem that $c^{(i)}$ is continuous on the real line.

Since the eight functions $f_{0,r}$, $f_{l,1}$, and $f_{l,r}^{(i)}$ for $i = 1, \dots, 6$ all vanish for all sufficiently large arguments, another application of the dominated convergence theorem shows that $c_0(x)$, $c_1(x)$, and $c^{(i)}(x)$ for $i = 1, \dots, 6$ all vanish in the limit as $x \rightarrow \infty$. This completes the proof. \square

Remark 5.12. For any $0 < t < 1$, by the fact that

$$J(t) \geq R_1(t) - L_1(t) \geq \min(t, 1 - t),$$

we have $\mathbb{P}(J(t) < \min(t, 1 - t)) = 0$ and thus $f_t(\min(t, (1 - t))) = 0$ by Theorem 5.11. This is a somewhat surprising result since we know that the right-continuous Dickman

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density f_0 satisfies $f_0(0) = e^{-\gamma} > 0$.

Remark 5.13. Since f_0 is both (uniformly) continuous on and piecewise differentiable on $(0, \infty)$, it might be natural to conjecture that the densities f_t for $0 < t < 1$ are also piecewise differentiable.

Chapter 6

Integral equation, positivity, and left- and right-tail behavior of f_t

6.1 Integral equation

In this section we prove that for $0 \leq t < 1$ and $x \in \mathbb{R}$, the density function $f_t(x)$ is jointly Borel measurable in (t, x) . By symmetry, we can conclude that $f_t(x)$ is jointly Borel measurable in (t, x) for $0 \leq t \leq 1$. We then use this result to establish an integral equation for the densities.

Let F_t denote the distribution function for $J(t)$. Because F_t is right continuous, it is Borel measurable (for each t).

Lemma 6.1. *For each positive integer n , the mapping*

$$(t, x) \mapsto F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) \quad (0 \leq t < 1, x \in \mathbb{R})$$

is Borel measurable.

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Proof. Note that

$$F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) = \sum_{j=1}^n \mathbf{1} \left(\frac{j-1}{n} \leq t < \frac{j}{n} \right) F_{\frac{j}{n}}(x).$$

Each term is the product of a Borel measurable function of t and a Borel measurable function of x and so is a Borel measurable of (t, x) . The same is then true of the sum. \square

Lemma 6.2. *For each $0 \leq t < 1$ and $x \in \mathbb{R}$, as $n \rightarrow \infty$ we have*

$$F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) \rightarrow F_t(x).$$

Proof. We reference Grübel and Rösler [21], who construct a process $J = (J(t))_{0 \leq t \leq 1}$ with $J(t)$ having distribution function F_t for each t and with right continuous sample paths. It follows (for each $t \in [0, 1)$) that F_u converges weakly to F_t as $u \downarrow t$. But we know that F_t is a continuous (and even continuously differentiable) distribution, so for each $x \in \mathbb{R}$ we have $F_u(x) \rightarrow F_t(x)$ as $u \downarrow t$. The result follows. \square

Proposition 6.3. *The mapping*

$$(t, x) \mapsto F_t(x) \quad (0 \leq t < 1, x \in \mathbb{R})$$

is Borel measurable.

Proof. According to Lemmas 6.1–6.2, this mapping is the pointwise limit as $n \rightarrow \infty$ of the Borel measurable mappings in Lemma 6.2. \square

Let f_t denote the continuous density for F_t , as in Theorem 5.11.

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Theorem 6.4. *The mapping*

$$(t, x) \mapsto f_t(x) \quad (0 \leq t < 1, x \in \mathbb{R})$$

is Borel measurable.

Proof. By the fundamental theorem of integral calculus, $f_t = F'_t$. The mapping in question is thus the (sequential) limit of difference quotients that are Borel measurable by Proposition 6.3 and hence is Borel measurable. \square

Now we are ready to derive integral equations. We start with an integral equation for the distribution functions F_t .

Proposition 6.5. *The distribution functions (F_t) satisfy the following integral equation for $0 \leq t \leq 1$ and $x \in \mathbb{R}$:*

$$F_t(x) = \int_{l \in (0,t)} F_{\frac{t-l}{1-l}} \left(\frac{x}{1-l} - 1 \right) dl + \int_{r \in (t,1)} F_{\frac{t}{r}} \left(\frac{x}{r} - 1 \right) dr. \quad (6.1)$$

Proof. This follows by conditioning on the value of $(L_1(t), R_1(t))$. Observe that each of the two integrands is (by Proposition 6.3 for $t \notin \{0, 1\}$ and by right continuity of F_0 and F_1 for $t \in \{0, 1\}$) indeed [for fixed (t, x)] a Borel measurable function of the integrating variable. \square

Remark 6.6. It follows from (i) the changes of variables from l to $v = (t-l)/(1-l)$ in the first integral in (6.1) and from r to $v = t/r$ in the second integral, (ii) the joint continuity of $f_t(x)$ in (t, x) established later in Corollary 7.8, and (iii) Leibniz's formula that $F_t(x)$ is differentiable with respect to $t \in (0, 1)$ for each fixed $x \in \mathbb{R}$.

Integral equation (6.1) for the distribution functions F_t immediately leads us to an integral equation for the density functions f_t .

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Proposition 6.7. *The continuous density functions (f_t) satisfy the following integral equation for $0 < t < 1$ and $x \in \mathbb{R}$:*

$$f_t(x) = \int_{l \in (0,t)} (1-l)^{-1} f_{\frac{t-l}{1-l}} \left(\frac{x}{1-l} - 1 \right) dl + \int_{r \in (t,1)} r^{-1} f_{\frac{t}{r}} \left(\frac{x}{r} - 1 \right) dr.$$

Proof. Fix $t \in (0, 1)$. Differentiate (6.1) with respect to x . It is easily proved by an argument applying the dominated convergence theorem to difference quotients and the mean value theorem that we can differentiate under the integral signs in (6.1) provided that

$$\int_{l \in (0,t)} (1-l)^{-1} f_{\frac{t-l}{1-l}}^* dl + \int_{r \in (t,1)} r^{-1} f_{\frac{t}{r}}^* dr \quad (6.2)$$

is finite, where f_t^* denotes any upper bound on $f_t(x)$ as x varies over \mathbb{R} . By Theorem 5.1 we can simply choose $f_t^* = 10$. Then (6.2) equals 10 times

$$-\ln(1-t) - \ln t,$$

which is finite. □

In the next proposition, we provide an integral equation based on the formula for f_t in (4.13). Recall that $Y(t) = \sum_{k=3}^{\infty} \Delta_k(t)$. Using (4.12), the conditional distribution of $Y(t)/(r_3 - l_3)$ given $(L_3, R_3) = (l_3, r_3)$ is the (unconditional) distribution of $Z(\frac{t-l_3}{r_3-l_3}) = 1 + J(\frac{t-l_3}{r_3-l_3})$. Apply Theorem 4.2 on $Z(\frac{t-l_3}{r_3-l_3})$ leads us to an integral equation for the density function of $J(t)$.

Proposition 6.8. *The continuous density functions f_t for the random variables $J(t) = Z(t) - 1$ satisfy the integral equation*

$$f_t(x) = \int \mathbb{P}((L_3(t), R_3(t)) \in d(l_3, r_3)) \cdot h_t(x \mid l_3, r_3)$$

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for $x \geq 0$, where

$$h_t(x \mid l_3, r_3) = \int f_{l_3, r_3}(x - y) (r_3 - l_3)^{-1} f_{\frac{t-l_3}{r_3-l_3}}\left(\frac{y}{r_3 - l_3} - 1\right) dy.$$

6.2 Positivity

In this section we prove that $f_t(x)$ is positive for every $x > \min\{t, 1 - t\}$.

Theorem 6.9. *For any $0 < t < 1$, the continuous density f_t satisfies*

$$f_t(x) > 0 \text{ if and only if } x > \min\{t, 1 - t\}.$$

We already know that $f_t(x) = 0$ if $x \leq \min\{t, 1 - t\}$, so we need only prove the “if” assertion. Our starting point for the proof is the following lemma. Recall from Chung [5, Exercise 1.6] that a point x is said to belong to the support of a distribution function F if for every $\epsilon > 0$ we have

$$F(x + \epsilon) - F(x - \epsilon) > 0. \tag{6.3}$$

Note that to prove that x is in the support of F we may choose any $\epsilon_0(x) > 0$ and establish (6.3) for all $\epsilon \in (0, \epsilon_0(x))$.

Lemma 6.10. *For any $0 < t < 1$, the support of F_t is $[\min\{t, 1 - t\}, \infty)$.*

Proof. Clearly the support of F_t is contained in $[\min\{t, 1 - t\}, \infty)$, so we need only establish the reverse containment. Since $F_t = F_{1-t}$ by symmetry, we may fix $t \leq 1/2$. Also fixing $x \geq t$, write

$$x = t + K + b$$

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where $K \geq 0$ is an integer and $b \in [0, 1)$. We will show that x belongs to the support of F_t . Let

$$A := \bigcap_{k=1}^K \{1 - k\epsilon < R_k < 1 - (k-1)\epsilon\}.$$

We break our analysis into four cases: (i) $t < b < 1$, (ii) $b = t$, (iii) $0 < b < t$, and (iv) $b = 0$.

(i) $t < b < 1$. Let

$$B := \{b < R_{K+1} < b + \epsilon\} \bigcap \{t < R_{K+2} < t + \epsilon\} \bigcap \{t - \epsilon < L_{K+3} < t\}$$

and

$$C := \left\{ 0 \leq \sum_{k=K+4}^{\infty} \Delta_k < 6\epsilon \right\}. \quad (6.4)$$

Upon observing that for $\delta_1, \delta_2 \in (0, \epsilon)$ we have by use of Markov's inequality that

$$\begin{aligned} & \mathbb{P}(C \mid (L_{K+3}, R_{K+3}) = (t - \delta_1, t + \delta_2)) \\ &= \mathbb{P}\left((\delta_1 + \delta_2)J\left(\frac{\delta_1}{\delta_1 + \delta_2}\right) < 6\epsilon\right) \geq \mathbb{P}\left(J\left(\frac{\delta_1}{\delta_1 + \delta_2}\right) < 3\right) \\ &\geq 1 - \frac{1}{3} \mathbb{E} J\left(\frac{\delta_1}{\delta_1 + \delta_2}\right) \geq 1 - \frac{1}{3} \left[1 + 2H\left(\frac{1}{2}\right)\right] > 0.2 > 0. \end{aligned} \quad (6.5)$$

We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ . But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^K (1 - k\epsilon) + b + t = x - \binom{K+1}{2} \epsilon$$

and

$$J(t) < \sum_{k=1}^K [1 - (k-1)\epsilon] + (b + \epsilon) + (t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 10\epsilon.$$

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We conclude that x is in the support of F_t .

(ii) $b = t$. Let

$$B := \{t < R_{K+2} < R_{K+1} < t + \epsilon\} \cap \{t - \epsilon < L_{K+3} < t\}$$

and define C by (6.4). We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ .

But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^K (1 - k\epsilon) + t + t = x - \binom{K+1}{2} \epsilon$$

and

$$J(t) < \sum_{k=1}^K [1 - (k-1)\epsilon] + 2(t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 10\epsilon.$$

We conclude that x is in the support of F_t .

(iii) $0 < b < t$. Let

$$B := \{t < R_{K+1} < t + \epsilon\} \cap \{t - b - \epsilon < L_{K+2} < t - b\} \cap \{t - \epsilon < L_{K+3} < t\}$$

and define C by (6.4). We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ .

But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^K (1 - k\epsilon) + t + b = x - \binom{K+1}{2} \epsilon$$

and

$$J(t) < \sum_{k=1}^K [1 - (k-1)\epsilon] + (t + \epsilon) + (b + 2\epsilon) + 2\epsilon + 6\epsilon \leq x + 11\epsilon.$$

We conclude that x is in the support of F_t .

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(iv) $b = 0$. Let

$$B := \{t < R_{K+1} < t + \epsilon\} \cap \{t - \epsilon < L_{K+2} < t\}$$

and define C by (6.4), but with $K + 4$ there changed to $K + 3$. We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small ϵ . But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^K (1 - k\epsilon) + t = x - \binom{K+1}{2} \epsilon$$

and

$$J(t) < \sum_{k=1}^K [1 - (k-1)\epsilon] + (t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 9\epsilon.$$

We conclude that x is in the support of F_t . □

We next use (5.6) together with Lemma 6.10 to establish Theorem 6.9 in a special case.

Lemma 6.11. *For any $0 < t < 1$, the continuous density f_t satisfies*

$$f_t(x) > 0 \text{ for all } x > 2 \min\{t, 1 - t\}.$$

Proof. We may fix $t \leq 1/2$ and $x > 2t$ and prove $f_t(x) > 0$. To do this, we first note

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from (5.6) that

$$\begin{aligned}
 f_t(x) &\geq c_t(x) = \mathbb{P}(L_2(t) = 0, J(t) \in dx)/dx \\
 &= \int_{r \in (t,1)} \int \mathbf{1}(r \leq x - rz < 1) (x - rz)^{-1} \mathbb{P}(Z(t/r) \in dz) dr \\
 &\geq \int_{r \in (t,1)} \int_{(x-1)/r}^{(x/r)-1} \mathbb{P}(Z(t/r) \in dz) dr \\
 &\geq \int_{r \in (t,1)} \mathbb{P}\left(\frac{x-1}{r} < Z\left(\frac{t}{r}\right) < \frac{x}{r} - 1\right) dr.
 \end{aligned}$$

According to Lemma 6.10, for the integrand in this last integral to be positive, it is necessary and sufficient that $(x-1)/r < (x/r) - 1$ (equivalently, $r < 1$) and

$$\frac{x}{r} - 1 > 1 + \min\left\{\frac{t}{r}, 1 - \frac{t}{r}\right\}$$

[for which it is sufficient that $r < (x+t)/3$]. Thus

$$f_t(x) \geq \int_{r \in (t, \min\{(x+t)/3, 1\})} \mathbb{P}\left(\frac{x-1}{r} < Z\left(\frac{t}{r}\right) < \frac{x}{r} - 1\right) dr > 0$$

because (recalling $x > 2t$) the integrand here is positive over the nondegenerate interval of integration. \square

Finally, we use a different contribution to $f_t(x)$ together with Lemma 6.11 to establish Theorem 6.9.

Proof of Theorem 6.9. We may fix $t \geq 1/2$ and $x > 1 - t$ and prove $f_t(x) > 0$. To do

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this, we first note that

$$\begin{aligned} f_t(x) &\geq \int_{l \in (0,t)} \int_{r \in (t,1)} \mathbb{P}(L_1(t) = L_2(t) \in dl, R_2(t) \in dr) \\ &\quad \left[\mathbb{P} \left((r-l)J \left(\frac{t-l}{r-l} \right) \in dx - [(1-l) + (r-l)] \right) / dx \right] \\ &= \int_{l \in (0,t)} \int_{r \in (t,1)} (1-l)^{-1} (r-l)^{-1} f_{\frac{t-l}{r-l}} \left(\frac{x - (1+r-2l)}{r-l} \right) dr dl \end{aligned}$$

According to Lemma 6.11, for the integrand in this double integral to be positive, it is sufficient that

$$\frac{x - (1+r-2l)}{r-l} > 2 \min \left\{ \frac{t-l}{r-l}, \frac{r-t}{r-l} \right\},$$

or, equivalently,

$$x > \min\{1 + 2t + r - 4l, 1 - 2t + 3r - 2l\}.$$

This strict inequality is true (because $x > 1 - t$) when $l = t$ and $r = t$ and so, for sufficiently small $\epsilon > 0$ is true for $l \in (t - \epsilon, t)$ and $r \in (t, t + \epsilon)$. Thus

$$f_t(x) \geq \int_{l \in (t-\epsilon, t)} \int_{r \in (t, t+\epsilon)} (1-l)^{-1} (r-l)^{-1} f_{\frac{t-l}{r-l}} \left(\frac{x - (1+r-2l)}{r-l} \right) dr dl > 0$$

because the integrand here is positive over the fully two-dimensional rectangle region of integration. □

6.3 Right-tail behavior

In this section we will prove, *uniformly* for $0 < t < 1$, that the continuous density functions f_t enjoy the same superexponential decay bound as Grübel and Rösler [21, Theorem 9] proved for the survival functions $1 - F_t$. By a separate and easier argu-

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ment, one could include the cases $t = 0, 1$. Let m_t denote the moment generating function of $Z(t)$ and recall that m denotes the moment generating function of V at (5.5). By Lemma 5.3, the random variables $Z(t)$, $0 \leq t \leq 1$, are stochastically dominated by V . As a consequence, if $\theta \geq 0$, then

$$m_t(\theta) \leq m(\theta) < \infty$$

for every $t \in (0, 1)$.

Theorem 6.12. *For any $0 < t < 1$, the continuous QuickQuant density $f_t(x)$ enjoys superexponential decay (uniformly in t) when x is large. More precisely, for any $\theta > 0$ we have*

$$f_t(x) < 4\theta^{-1}e^{2\theta}m(\theta)e^{-\theta x}$$

for $x \geq 3$, where m is the moment generating function of the random variable V at (5.5).

Proof. Our starting point is the following equation from the discussion preceding Proposition 6.8:

$$\begin{aligned} f_t(x) &= \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) \int_y f_{l,r}(x-y) \mathbb{P}\left((r-l)Z\left(\frac{t-l}{r-l}\right) \in dy\right) \\ &= \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) \int_z f_{l,r}(x-(r-l)z) \mathbb{P}\left(Z\left(\frac{t-l}{r-l}\right) \in dz\right). \end{aligned} \tag{6.6}$$

By Lemma 5.3, for any $\theta \in \mathbb{R}$ we can obtain a probability measure $\mu_{t,\theta}(dz) := m_t(\theta)^{-1}e^{\theta z} \mathbb{P}(Z(t) \in dz)$ by exponential tilting. Since $m_t(\theta) \leq m(\theta) < \infty$ for every

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$\theta \geq 0$ and $t \in (0, 1)$, we can rewrite and bound (6.6) as follows (for any $\theta \geq 0$):

$$\begin{aligned} f_t(x) &= \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) m_{\frac{t-l}{r-l}}(\theta) \int_z e^{-\theta z} f_{l,r}(x - (r-l)z) \mu_{\frac{t-l}{r-l}, \theta}(dz) \\ &\leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) \int_z e^{-\theta z} f_{l,r}(x - (r-l)z) \mu_{\frac{t-l}{r-l}, \theta}(dz). \end{aligned}$$

Recall that $f_{l,r}(x)$ is bounded above by $b_t(l, r)$ (Lemmas 5.2 and 5.8–5.9) and vanishes for $x \geq 2$. Therefore, if $\theta \geq 0$ then

$$\begin{aligned} f_t(x) &\leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) b_t(l, r) \int_{z > \frac{x-2}{r-l}} e^{-\theta z} \mu_{\frac{t-l}{r-l}, \theta}(dz) \\ &\leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) b_t(l, r) \exp\left(-\theta \frac{x-2}{r-l}\right). \end{aligned} \quad (6.7)$$

Suppose $x \geq 3$ and $\theta > 0$. We now consider in turn the contribution to (6.7) for $l = 0$, for $r = 1$, and for $0 < l < t < r < 1$. For $l = 0$, the contribution is $m(\theta)$ times the following:

$$\begin{aligned} &\int_t^1 r^{-1} \beta \exp[-\theta r^{-1}(x-2)] dr \\ &\leq \beta \int_0^1 r^{-2} \exp[-\theta r^{-1}(x-2)] dr \\ &= \beta [\theta(x-2)]^{-1} \exp[-\theta(x-2)] \leq \beta \theta^{-1} e^{2\theta} e^{-\theta x}. \end{aligned}$$

Similarly (or symmetrically), the contribution for $r = 1$ is bounded by the same $\beta \theta^{-1} e^{2\theta} m(\theta) e^{-\theta x}$. For $0 < l < t < r < 1$, by symmetry we may without loss of generality suppose that $0 < t \leq 1/2$, and then the contribution is $\frac{3}{2}m(\theta)$ times the

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following:

$$\begin{aligned}
& \int_0^t \int_t^1 \left[\frac{1}{r(r-l)} + \frac{1}{(1-l)(r-l)} \right] \exp \left(-\theta \frac{x-2}{r-l} \right) dr dl \\
&= \int_0^t \int_{t-l}^{1-l} \left[\frac{1}{(s+l)s} + \frac{1}{(1-l)s} \right] \exp[-\theta s^{-1}(x-2)] ds dl \\
&= \int_0^t \int_{t-l}^{1-l} (1-l)^{-1} (s+l)^{-1} (1+s) s^{-1} \exp[-\theta s^{-1}(x-2)] ds dl \\
&\leq 4 \int_0^t \int_{t-l}^{1-l} s^{-2} \exp[-\theta s^{-1}(x-2)] ds dl \\
&\leq 4 \int_0^{1/2} \int_0^1 s^{-2} \exp[-\theta s^{-1}(x-2)] ds dl \\
&= 2[\theta(x-2)]^{-1} \exp[-\theta(x-2)] \leq 2\theta^{-1} e^{2\theta} e^{-\theta x}.
\end{aligned}$$

Summing all the contributions, we find

$$f_t(x) \leq (3 + 2\beta)\theta^{-1} e^{2\theta} m(\theta) e^{-\theta x} < 4\theta^{-1} e^{2\theta} m(\theta) e^{-\theta x}, \quad (6.8)$$

for any $0 < t < 1$, $x \geq 3$, and $\theta > 0$, demonstrating the uniform superexponential decay. \square

Remark 6.13. Since f_t is uniformly bounded by 10 by Theorem 5.1, for any $\theta > 0$, by choosing the coefficient $C_\theta := \max\{10e^{3\theta}, 4\theta^{-1}e^{2\theta}m(\theta)\}$, we can extend the superexponential bound on $f_t(x)$ in Theorem 6.12 for $x \geq 3$ to $x \in \mathbb{R}$ as

$$f_t(x) \leq C_\theta e^{-\theta x} \text{ for } x \in \mathbb{R} \text{ and } 0 < t < 1. \quad (6.9)$$

Note that this bound is not informative for $x \leq \min\{t, 1-t\}$ since we know $f_t(x) = 0$ for such x (by Theorem 6.9), but it will simplify our proof of Theorem 7.1.

6.4 Left-tail behavior

We consider the densities f_t with $t \in (0, 1)$; since $f_t \equiv f_{1-t}$ by symmetry, we may without loss of generality suppose $t \in (0, 1/2]$. As previously noted (recall Theorems 6.9 and 5.11), $f_t(x) = 0$ for all $x \leq t$ and $f_t(x) > 0$ for all $x > t$. In this section we consider the left-tail behavior of f_t , by which we mean the behavior of $f_t(x)$ as $x \downarrow t$.

As a warm-up, we first show that f_t has a positive right-hand derivative at t that is large when t is small.

Lemma 6.14.

(a) *Fix $t \in (0, 1/2)$. Then the density function f_t has right-hand derivative $f'_t(t)$ at t equal to c_1/t , where*

$$c_1 := \int_0^1 \mathbb{E}[2 - w + J(w)]^{-2} dw \in (0.0879, 0.3750).$$

(b) *Fix $t = 1/2$. Then the density function f_t has right-hand derivative $f'_t(t)$ at t equal to $2c_1/t = 4c_1$.*

Proof. (a) We begin with two key observations. First, if $L_1(t) > 0$, then $J(t) > 1 - t$. Second, if $1 > R_1(t) > R_2(t)$, then $J(t) > 2t$. It follows that if $0 < z < \min\{1 - 2t, t\}$,

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then, with $Y \equiv Y(t)$ as defined at (4.12),

$$\begin{aligned}
& f_t(t+z) \, dz \\
&= \mathbb{P}(J(t) - t \in dz) \\
&= \mathbb{P}(R_1(t) < 1, L_2(t) > 0, J(t) - t \in dz) \\
&= \iint_{\substack{y>x>0: x+y<z, \\ x<1-t, y-x<t}} \mathbb{P}(R_1(t) - t \in dx, t+x-L_2(t) \in dy, Y(t) \in dz-x-y) \\
&= \iint_{\substack{y>x>0: x+y<z, \\ x<1-t, y-x<t}} dx \frac{dy}{t+x} \mathbb{P}\left(y J\left(\frac{y-x}{y}\right) \in dz-x-y\right) \\
&= \iint_{\substack{y>x>0: x+y<z, \\ x<1-t, y-x<t}} dx \frac{dy}{t+x} y^{-1} f_{1-\frac{x}{y}}\left(\frac{z-x-y}{y}\right) dz.
\end{aligned}$$

Now make the changes of variables from x to $u = x/z$ and from y to $v = y/z$. We then find

$$\begin{aligned}
f_t(t+z) &= z \iint_{\substack{v>u>0: u+v<1 \\ u<(1-t)/z, v-u<t/z}} (t+uz)^{-1} v^{-1} f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) du dv \\
&= z \iint_{v>u>0: u+v<1} (t+uz)^{-1} v^{-1} f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) du dv,
\end{aligned}$$

where the second equality follows because $(1-t)/z > (1-2t)/z > 1$ and $t/z > 1$ by assumption. Thus, as desired,

$$f_t(t+z) \sim \frac{c_1 z}{t}$$

as $z \downarrow 0$ by the dominated convergence theorem, if we can show that

$$\tilde{c} := \iint_{v>u>0: u+v<1} v^{-1} f_{1-\frac{u}{v}}\left(\frac{1-u-v}{v}\right) du dv$$

equals c_1 . For that, make another change of variables from u to $w = u/v$; then we

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find

$$\begin{aligned}\tilde{c} &= \int_0^1 \int_0^{(1+w)^{-1}} f_{1-w}(v^{-1} - (1+w)) \, dv \, dw \\ &= \int_0^1 \int_0^{(2-w)^{-1}} f_w(v^{-1} + w - 2) \, dv \, dw.\end{aligned}$$

Make one last change of variables, from v to $s = v^{-1} + w - 2$, to conclude

$$\tilde{c} = \int_0^1 \int_0^\infty (2 - w + s)^{-2} f_w(s) \, ds \, dw = c_1,$$

as claimed.

To obtain the claimed upper bound on c_1 , we note, using the facts that $J(w)$ and $J(1-w)$ have the same distribution and that $J(w) > w$ for $w \in (0, 1/2)$, that

$$\begin{aligned}c_1 &= \int_0^{1/2} \mathbb{E}[2 - w + J(w)]^{-2} \, dw + \int_{1/2}^1 \mathbb{E}[2 - w + J(w)]^{-2} \, dw \\ &= \int_0^{1/2} \mathbb{E}[2 - w + J(w)]^{-2} \, dw + \int_0^{1/2} \mathbb{E}[1 + w + J(w)]^{-2} \, dw \\ &< \int_0^{1/2} \frac{1}{4} \, dw + \int_0^{1/2} (1 + 2w)^{-2} \, dw = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} = 0.375.\end{aligned}$$

To obtain the claimed lower bound on c_1 , we combine Jensen's inequality with the known fact [cf. (4.2)] that $\mathbb{E} J(w) = 1 + 2H(w)$ with $H(w) = -w \ln w - (1-w) \ln(1-w)$:

$$\begin{aligned}c_1 &= \int_0^1 \mathbb{E}[2 - w + J(w)]^{-2} \, dw \\ &\geq \int_0^1 (\mathbb{E}[2 - w + J(w)])^{-2} \, dw = \int_0^1 (3 - w + 2H(w))^{-2} \, dw > 0.0879.\end{aligned}$$

(b) By an argument similar to that at the start of the proof of (a), if $0 < z < 1/2$,

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then, using symmetry at the third equality,

$$\begin{aligned}
 f_t(t+z) dz &= \mathbb{P}(J(t) - t \in dz) \\
 &= \mathbb{P}(R_1(t) < 1, L_2(t) > 0, J(t) - t \in dz) \\
 &\quad + \mathbb{P}(L_1(t) > 0, R_2(t) < 1, J(t) - t \in dz) \\
 &= 2 \mathbb{P}(R_1(t) < 1, L_2(t) > 0, J(t) - t \in dz) \\
 &\sim \frac{2c_1 z}{t} = 4c_1 z.
 \end{aligned}$$

Here the asymptotic equivalence is as $z \downarrow 0$ and follows by the same argument as used for (a). \square

We are now prepared for our main result about the left-tail behavior of f_t .

Theorem 6.15.

(a) *Fix $t \in (0, 1/2)$. Then $f_t(t+tz)$ has the uniformly absolutely convergent power series expansion*

$$f_t(t+tz) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k$$

for $z \in [0, \min\{t^{-1} - 2, 1\})$, where for $k \geq 1$ the coefficients

$$c_k := \int_0^1 (1-w)^{k-1} \mathbb{E}[2-w+J(w)]^{-(k+1)} dw,$$

not depending on t , are strictly positive, have the property that $2^k c_k$ is strictly decreasing in k , and satisfy

$$0 < (0.0007)2^{-(k+1)}(k+1)^{-2} < c_k < 2^{-(k+1)}k^{-1}(1+2^{-k}) < 0.375 < \infty.$$

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[In particular, $2^k c_k$ is both $O(k^{-1})$ and $\Omega(k^{-2})$.]

(b) Fix $t = 1/2$. Then $f_t(t + tz)$ has the uniformly absolutely convergent power series expansion

$$f_t(t + tz) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k$$

for $z \in [0, 1)$.

Proof. (a) As shown in the proof of Lemma 6.14, for $z \in [0, \min\{t^{-1} - 2, 1\})$ we have

$$f_t(t + tz) = z \iint_{v>u>0: u+v<1} (1 + uz)^{-1} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1 - u - v}{v} \right) du dv. \quad (6.10)$$

Note that the expression on the right here doesn't depend on t . Further, since $z \leq 1$ and $0 < u < 1/2$ in the range of integration,

$$\begin{aligned} & \frac{1}{2} \iint_{v>u>0: u+v<1} (1 - uz)^{-1} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1 - u - v}{v} \right) du dv \\ & < \iint_{v>u>0: u+v<1} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1 - u - v}{v} \right) du dv \\ & = \tilde{c} = c_1 < 3/8 < \infty, \end{aligned}$$

with \tilde{c} and c_1 as in the proof of Lemma 6.14. It follows that $f_t(t + tz)$ has the uniformly absolutely convergent power series expansion

$$f_t(t + tz) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k$$

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for $z \in [0, \min\{t^{-1} - 2, 1\})$, where for $k \geq 1$ we have

$$\begin{aligned} c_k &= 2 \times 2^{-k} \iint_{v>u>0: u+v<1} (2u)^{k-1} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1-u-v}{v} \right) du dv \\ &= \int_0^1 (1-w)^{k-1} \mathbb{E}[2-w+J(w)]^{-(k+1)} dw; \end{aligned}$$

the second equality follows just as for $c = c_1$ in the proof of Lemma 6.14. From the first equality it is clear that these coefficients have the property that $2^k c_k$ is strictly decreasing in k .

To obtain the claimed upper bound on c_k , proceed just as in the proof of Lemma 6.14 to obtain

$$\begin{aligned} c_k &< 2^{-(k+1)} \int_0^{1/2} (1-w)^{k-1} dw + \int_0^{1/2} w^{k-1} (1+2w)^{-(k+1)} dw \\ &= 2^{-(k+1)} k^{-1} (1-2^{-k}) + k^{-1} 4^{-k} = 2^{-(k+1)} k^{-1} (1+2^{-k}). \end{aligned}$$

The claimed lower bound on c_k follows from Lemma 6.14 for $k = 1$ but for $k \geq 2$ requires more work. We begin by establishing a lower bound on $\mathbb{P}(J(w) \leq 2w)$ for $w \leq 1/3$, using what we have already proved:

$$\begin{aligned} \mathbb{P}(J(w) \leq 2w) &= \int_0^w f_w(w+x) dx = w \int_0^1 f_w(w+wz) dz \\ &\geq w \int_0^1 (c_1 z - c_2 z^2) dz = w(\tfrac{1}{2}c_1 - \tfrac{1}{3}c_2) \\ &> [0.04395 - (1/3)(1/8)(1/2)(5/4)]w > 0.0179w. \end{aligned}$$

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Thus c_k is at least $0.0179 \cdot 2^{-(k+1)}$ times the following expression:

$$\begin{aligned}
& 2^{k+1} \int_0^{1/3} w(1-w)^{k-1}(2+w)^{-(k+1)} dw \\
& \geq \int_0^{1/3} w \exp[-2(k+1)w] \exp[-(k+1)w/2] dw \\
& \geq \int_0^{1/(k+1)} w \exp[-5(k+1)w/2] dw \\
& \geq e^{-5/2} \int_0^{1/(k+1)} w dw = \frac{1}{2} e^{-5/2} (k+1)^{-2}.
\end{aligned}$$

(b) The claim of part (b) is clear from the proof of Lemma 6.14. \square

Corollary 6.16.

(a) Fix $t \in (0, 1/2)$. Then, for all $x \in (t, \min\{1-t, 2t\})$, the density $f_t(x)$ is infinitely differentiable, strictly increasing, strictly concave, and strictly log-concave.

(b) Fix $t = 1/2$. Then for all $x \in [1/2, 1)$, the density $f_{1/2}(x)$ is infinitely differentiable, strictly increasing, strictly concave, and strictly log-concave.

Proof. Once again it is clear that we need only prove (a). The result is actually a corollary to (6.10), rather than to Theorem 6.15. It is easy to justify repeated differentiation with respect to z under the double integral of (6.10). In particular, for $z \in (0, \min\{t^{-1} - 2, 1\})$ we have

$$\begin{aligned}
t f'_t(t + tz) &= \iint_{v>u>0: u+v<1} (1+uz)^{-1} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1-u-v}{v} \right) du dv \\
&\quad - z \iint_{v>u>0: u+v<1} u(1+uz)^{-2} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1-u-v}{v} \right) du dv \\
&= \iint_{v>u>0: u+v<1} (1+uz)^{-2} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1-u-v}{v} \right) du dv > 0
\end{aligned}$$

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and

$$t^2 f_t''(t + tz) = -2 \iint_{v > u > 0: u+v < 1} u(1 + uz)^{-3} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1 - u - v}{v} \right) du dv$$

$$< 0.$$

Strict log-concavity of the positive function f_t follows immediately from strict concavity. □

Remark 6.17. (a) By extending the computations of the first and second derivatives of f_t in the proof of Corollary 6.16 to higher-order derivatives, it is easy to see that $f_t(x)$ is real-analytic for x in the intervals as specified in Corollary 6.16(a)–(b). For the definition of real analytic function, see Krantz and Parks [28, Definition 1.1.5].

(b) It may be that, like the Dickman density f_0 , the densities f_t with $0 < t < 1$ are log-concave everywhere and hence strongly unimodal. Even if this is false, we conjecture that the densities f_t are all unimodal.

Chapter 7

Lipschitz continuity of f_t

7.1 Lipschitz continuity

We now prove that, for each $0 < t < 1$, the density function f_t is Lipschitz continuous, which is a result stronger than Theorem 5.11.

Theorem 7.1. *For each $0 < t < 1$, the density function f_t is Lipschitz continuous.*

That is, there exists a constant $\Lambda_t \in (0, \infty)$ such that for any $x, z \in \mathbb{R}$, we have $|f_t(z) - f_t(x)| \leq \Lambda_t |z - x|$. The proof of Theorem 7.1 will reveal that one can take $\Lambda_t = \Lambda[t^{-1} \ln t][(1 - t)^{-1} \ln(1 - t)]$ for some constant $\Lambda < \infty$. Thus the densities f_t are in fact uniformly Lipschitz continuous for t in any compact subinterval of $(0, 1)$.

We break the proof of Theorem 7.1 into two lemmas. Lemma 7.2 deals with the contribution to f_t from the disjoint-union event $\{0 = L_3(t) < t < R_3(t) < 1\} \cup \{0 < L_3(t) < t < R_3(t) = 1\}$ while Lemma 7.3 deals with the contribution from the event $\{0 < L_3(t) < t < R_3(t) < 1\}$.

Lemma 7.2. *For each $0 < t < 1$, the contribution to f_t from the event $\{0 = L_3(t) < t < R_3(t) < 1\} \cup \{0 < L_3(t) < t < R_3(t) = 1\}$ is Lipschitz continuous.*

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Proof. Fix $0 < t < 1$. By symmetry, we need only consider the contribution to $f_t(x)$ from the event $\{0 = L_3(t) < t < R_3(t) < 1\}$. Recall that this contribution is

$$c_0(x) := \frac{1}{2} \int_{r,y} (\ln r)^2 f_{0,r}(x-y) \mathbb{P}(Y \in dy \mid L_3(t) = 0, R_3(t) = r) dr,$$

and that the conditional probability in the integrand can be written as

$$\mathbb{P}(Y \in dy \mid L_3(t) = 0, R_3(t) = r) = \frac{1}{r} f_{\frac{t}{r}}\left(\frac{y}{r} - 1\right) dy.$$

Let $z, x \in \mathbb{R}$ with $z > x$ and fixed $r \in (t, 1)$. Writing

$$d_r(x, z, y) := \frac{1}{2} (\ln r)^2 [f_{0,r}(z-y) - f_{0,r}(x-y)],$$

we are interested in bounding the absolute difference

$$|c_0(z) - c_0(x)| \leq \int_{r,y} |d_r(x, z, y)| \frac{1}{r} f_{\frac{t}{r}}\left(\frac{y}{r} - 1\right) dy.$$

Case 1. $z - x \leq 1 - r$. We bound $d_r(x, z, y)$ for y in each of the seven subintervals of the real line determined by the six partition points

$$x - 2 < z - 2 \leq x - (1 + r) < z - (1 + r) \leq x - 2r < z - 2r,$$

and then the contribution to our bound on $|c_0(z) - c_0(x)|$ from all y in that subinterval (and all r satisfying the restriction of Case 1). For the two subcases $y \leq x - 2$ and $y > z - 2r$, we have $d_r(x, z, y) = 0$. We bound the five nontrivial subcases (not listed in natural order as the subintervals) as follows.

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Subcase 1(a). $x - 2 < y \leq z - 2$. We have

$$|d_r(x, z, y)| = \left| \frac{1}{x - y} \right| \ln \left(\frac{1}{x - y - 1} \right) \leq \frac{1}{1 + r} \ln \frac{1}{r},$$

and the contribution to $|c_0(z) - c_0(x)|$ is bounded by

$$\int_{r=t}^1 \frac{1}{r(1+r)} \left(\ln \frac{1}{r} \right) \int_{y=x-2}^{z-2} f_{\frac{t}{r}} \left(\frac{y}{r} - 1 \right) dy dr \leq 10(z-x) \frac{1-t}{t(1+t)} \ln \frac{1}{t}$$

since $f_{t/r}$ is bounded by 10.

Subcase 1(b). $z - 2 < y \leq x - (1 + r)$. We have

$$\begin{aligned} d_r(x, z, y) &= \frac{1}{z - y} \ln \left(\frac{1}{z - y - 1} \right) - \frac{1}{x - y} \ln \left(\frac{1}{x - y - 1} \right) \\ &= \frac{1}{z - y} \left[\ln \left(\frac{1}{z - y - 1} \right) - \ln \left(\frac{1}{x - y - 1} \right) \right] \\ &\quad + \left(\frac{1}{z - y} - \frac{1}{x - y} \right) \ln \left(\frac{1}{x - y - 1} \right). \end{aligned}$$

Observe that $z - y > x - y > 1 + r$ and that the function $\ln[1/(x - 1)]$ is differentiable for $x > 1$. We then use the mean value theorem to obtain

$$\begin{aligned} |d_r(x, z, y)| &\leq \frac{1}{1 + r} \left| \ln \left(\frac{1}{z - y - 1} \right) - \ln \left(\frac{1}{x - y - 1} \right) \right| + \frac{(z - x)}{(1 + r)^2} \ln \frac{1}{r} \\ &\leq (z - x) \left[\frac{1}{r(1 + r)} + \frac{1}{(1 + r)^2} \ln \frac{1}{r} \right]. \end{aligned}$$

The contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$(z - x) \int_{r=t}^1 \left[\frac{1}{r(1 + r)} + \frac{1}{(1 + r)^2} \ln \frac{1}{r} \right] dr \leq (z - x) \frac{1 - t}{1 + t} \left(\frac{1}{t} + \frac{1}{1 + t} \ln \frac{1}{t} \right).$$

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Subcase 1(c). $x - (1 + r) < y \leq z - (1 + r)$. We have

$$\begin{aligned} d_r(x, z, y) &= \frac{1}{z - y} \ln \left(\frac{1}{z - y - 1} \right) - \frac{1}{x - y} \ln \left(\frac{x - y - r}{r} \right) \\ &= \frac{1}{z - y} \left[\ln \left(\frac{1}{z - y - 1} \right) - \ln \left(\frac{x - y - r}{r} \right) \right] \\ &\quad + \left(\frac{1}{z - y} - \frac{1}{x - y} \right) \ln \left(\frac{x - y - r}{r} \right). \end{aligned}$$

Using the inequalities $z - y \geq 1 + r$ and $x - y > 2r$, we have

$$|d_r(x, z, y)| = \frac{1}{1 + r} \left| \ln \left(\frac{1}{z - y - 1} \right) - \ln \left(\frac{x - y - r}{r} \right) \right| + \frac{z - x}{2r(1 + r)} \ln \frac{1}{r}.$$

We can bound the absolute-value term here by

$$\begin{aligned} &\left| \ln \frac{1}{z - y - 1} - \ln \frac{1}{(1 + r) - 1} \right| + \left| \ln \frac{(1 + r) - r}{r} - \ln \frac{x - y - r}{r} \right| \\ &\leq \frac{1}{r} [z - y - (1 + r)] + \frac{1}{r} [(1 + r) - (x - y)] = (z - x) \frac{1}{r}, \end{aligned}$$

where the above inequality comes from two applications of the mean value theorem.

The contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$(z - x) \int_{r=t}^1 \left[\frac{1}{r(1 + r)} + \frac{1}{2r(1 + r)} \ln \frac{1}{r} \right] dr \leq (z - x) \frac{1 - t}{t(1 + t)} \left(1 + \frac{1}{2} \ln \frac{1}{t} \right).$$

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Subcase 1(d). $z - (1 + r) < y \leq x - 2r$. We have

$$\begin{aligned} d_r(x, z, y) &= \frac{1}{z-y} \ln \left(\frac{z-y-r}{r} \right) - \frac{1}{x-y} \ln \left(\frac{x-y-r}{r} \right) \\ &= \frac{1}{z-y} \left[\ln \left(\frac{z-y-r}{r} \right) - \ln \left(\frac{x-y-r}{r} \right) \right] \\ &\quad + \left(\frac{1}{z-y} - \frac{1}{x-y} \right) \ln \left(\frac{x-y-r}{r} \right). \end{aligned}$$

Using the inequality $z - y > x - y \geq 2r$, we obtain

$$\begin{aligned} |d_r(x, z, y)| &\leq \frac{1}{2r} [\ln(z-y-r) - \ln(x-y-r)] + \frac{z-x}{(2r)^2} \ln \frac{1}{r} \\ &\leq (z-x) \left[\frac{1}{2r} \frac{1}{r} + \frac{1}{(2r)^2} \ln \frac{1}{r} \right] \end{aligned}$$

by the differentiability of $\ln(x-r)$ for $x > r$ and the mean value theorem. The contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$(z-x) \int_{r=t}^1 \left[\frac{1}{2r} \frac{1}{r} + \frac{1}{(2r)^2} \ln \frac{1}{r} \right] dr = (z-x) \frac{(1-t) + \ln(1/t)}{4t}.$$

Subcase 1(e). $x - 2r < y \leq z - 2r$. Using the inequality $2r \leq z - y < 1 + r$, we have

$$|d_r(x, z, y)| = \frac{1}{z-y} \ln \left(\frac{z-y-r}{r} \right) \leq \frac{1}{2r} \ln \frac{1}{r},$$

and the contribution to $|c_0(z) - c_0(x)|$ is then bounded by

$$\int_{r=t}^1 \frac{1}{2r^2} \left(\ln \frac{1}{r} \right) \int_{y=x-2r}^{z-2r} f_{\frac{t}{r}} \left(\frac{y}{r} - 1 \right) dy dr \leq 10(z-x) \frac{1-t}{2t^2} \ln \frac{1}{t}.$$

This completes the proof for Case 1.

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Case 2. $z - x > 1 - r$. We directly bound

$$|d_r(x, z, y)| \leq \frac{1}{2}(\ln r)^2[f_{0,r}(z - y) + f_{0,r}(x - y)].$$

If $z - x \leq 1 - t$, use the bound in Remark 5.10; we can then bound the contribution to $|c_0(z) - c_0(x)|$ by

$$\int_{r=1-(z-x)}^1 \frac{2\beta}{r} dr \leq (z - x) \frac{2\beta}{t}.$$

On the other hand, if $z - x > 1 - t$, then we can bound the contribution to $|c_0(z) - c_0(x)|$ by

$$\frac{z - x}{1 - t} \int_{r=t}^1 \frac{2\beta}{r} dr \leq (z - x) \frac{2\beta}{t}.$$

This completes the proof for Case 2. We conclude that c_0 is a Lipschitz continuous function; note that the Lipschitz constant we have obtained depends on t . \square

Lemma 7.3. *For each $0 < t < 1$, the contribution to f_t from the event $\{0 < L_3(t) < t < R_3(t) < 1\}$ is Lipschitz continuous.*

Proof. Fix $0 < t < 1$. According to (4.25) and (4.27), the contribution from the event in question to $f_t(x)$ is $\sum_{i=1}^6 c^{(i)}(x)$, where we define

$$c^{(i)}(x) := \int_{l,r,y} f_{l,r}^{(i)}(x - y) \mathbb{P}(Y \in dy \mid (L_3(t), R_3(t)) = (l, r)) dl dr.$$

We show here that $c^{(3)}$ is Lipschitz continuous, and the claims that the other contributions $c^{(i)}$ are Lipschitz continuous are proved similarly.

Let $x, z \in \mathbb{R}$ with $z > x$ and consider (l, r) satisfying $0 < l < t < r < 1$. Define

$$d_{l,r}(x, z, y) := f_{l,r}^{(3)}(z - y) - f_{l,r}^{(3)}(x - y)$$

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and reformulate

$$f_{l,r}^{(3)}(x) = \mathbb{1}(1+r-2l \leq x < 1+r) \frac{1}{x} \left(\frac{1}{x+1-r} + \frac{1}{x+r-1} \right)$$

from the expression for $f_{l,r}^{(3)}(x)$ found in Section 4.2. We are interested in bounding the quantity

$$|c^{(3)}(z) - c^{(3)}(x)| \leq \int_{l,r,y} |d_{l,r}(x, z, y)| \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) dl dr, \quad (7.1)$$

where the conditional probability can also be written in density terms as

$$\mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) = \frac{1}{r-l} f_{\frac{r-l}{r-l}} \left(\frac{y}{r-l} - 1 \right) dy.$$

Just as we did for Lemma 7.2, we break the proof into consideration of two cases.

Case 1. $z - x < 2l$. As in the proof for Case 1 of Lemma 7.2, we bound $d_{l,r}(x, z, y)$ for y in each of the five subintervals of the real line determined by the four partition points

$$x - (1+r) < z - (1+r) < x - (1+r-2l) < z - (1+r-2l).$$

For the two subcases $y \leq x - (1+r)$ and $y > z - (1+r-2l)$, we have $d_r(x, z, y) = 0$.

We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

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Subcase 1(a). $z - (1 + r) < y \leq x - (1 + r - 2l)$. We have

$$\begin{aligned}
 d_{l,r}(x, z, y) &= \frac{1}{z - y} \left(\frac{1}{z - y + 1 - r} - \frac{1}{z - y + r - 1} \right) \\
 &\quad - \frac{1}{x - y} \left(\frac{1}{x - y + 1 - r} - \frac{1}{x - y + r - 1} \right) \\
 &= \frac{1}{z - y} \left(\frac{1}{z - y + 1 - r} - \frac{1}{x - y + 1 - r} \right) + \left(\frac{1}{z - y} - \frac{1}{x - y} \right) \frac{1}{x - y + 1 - r} \\
 &\quad - \frac{1}{z - y} \left(\frac{1}{z - y + r - 1} - \frac{1}{x - y + r - 1} \right) - \left(\frac{1}{z - y} - \frac{1}{x - y} \right) \frac{1}{x - y + r - 1}.
 \end{aligned}$$

Using the inequality $z - y > x - y \geq 1 + r - 2l$, we obtain

$$\begin{aligned}
 |d_{l,r}(x, z, y)| &\leq \frac{1}{1 + r - 2l} \frac{z - x}{(2 - 2l)^2} + \frac{z - x}{(1 + r - 2l)^2} \frac{1}{2 - 2l} \\
 &\quad + \frac{1}{1 + r - 2l} \frac{z - x}{(z - y + r - 1)(x - y + r - 1)} + \frac{z - x}{(1 + r - 2l)^2} \frac{1}{2(r - l)}.
 \end{aligned}$$

Except for the third term, it is easy to see (by direct computation) that the corresponding contribution to the bound (7.1) on $|c^{(3)}(z) - c^{(3)}(x)|$ is bounded by a constant (depending on t) times $z - x$. So we now focus on bounding the contribution from the third term. Note that since $1 + r - 2l > 1 - t > 0$, we need only bound

$$\int_{l,r,y} \frac{1}{(z - y + r - 1)(x - y + r - 1)} \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) dl dr \quad (7.2)$$

by a constant (which is allowed to depend on t , but our constant will not).

We first focus on the integral in (7.2) with respect to y and write it, using a change

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of variables, as

$$\int_{y \in I} d_{l,r}^*(x, z, y) f_{\frac{t-l}{r-l}}(y) dy, \quad (7.3)$$

with

$$d_{l,r}^*(x, z, y) = \frac{1}{[z - (r-l)(y+1) + r-1][x - (r-l)(y+1) + r-1]}$$

and $I := \left\{ y : \frac{z-(1+r)}{r-l} - 1 < y \leq \frac{x-(1+r-2l)}{r-l} - 1 \right\}$. Because the support of the density $f_{\frac{t-l}{r-l}}$ is contained in the nonnegative real line, the integral (7.3) vanishes unless the right endpoint of the interval I is positive, which is true if and only if

$$r < \frac{x-1+3l}{2}.$$

So we see that the integral of (7.3) over $r \in (t, 1)$ vanishes unless this upper bound on r is larger than t , which is true if and only if

$$l > \frac{1-x+2t}{3}. \quad (7.4)$$

But then the integral of (7.3) over $\{(l, r) : 0 < l < t < r < 1\}$ vanishes unless this lower bound on l is smaller than t , which is true if and only if $x > 1-t$; we conclude that for $x \leq 1-t$, that integral vanishes.

So we may now suppose $x > 1-t$, and we have seen that the integral of (7.3) over $\{(l, r) : 0 < l < t < r < 1\}$ is bounded above by its integral over the region

$$R := \left\{ (l, r) : \frac{1-x+2t}{3} \vee 0 < l < t < r < 1 \wedge \frac{x-1+3l}{2} \right\}.$$

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Observe that on R we have

$$\frac{x - (1 + r - 2l)}{r - l} - 1 = \frac{x - 1 + l}{r - l} - 2 > \frac{2}{3} \frac{x + t - 1}{r - l} - 2 > \frac{1}{2} \frac{x + t - 1}{r - l} - 2. \quad (7.5)$$

Define

$$B := \left\{ (l, r) : \frac{x + t - 1}{2(r - l)} - 2 > 0 \right\}.$$

We now split our discussion of the contribution to the integral of (7.3) over $(l, r) \in R$ into two terms, corresponding to (i) $R \cap B^c$ and (ii) $R \cap B$.

Term (i). $R \cap B^c$. Using (7.5), we can bound (7.3) by extending the range of integration from I to

$$I^* := \left\{ y : \frac{x + t - 1}{2(r - l)} - 2 < y \leq \frac{x - (1 + r - 2l)}{r - l} - 1 \right\}.$$

Making use of the inequality (6.9), the integral (7.3) is bounded, for any $\theta > 0$, by

$$\int_{y \in I^*} \frac{1}{4(r - l)^2} C_\theta e^{-\theta y} dy \leq \frac{C_\theta}{4\theta(r - l)^2} \exp \left[-\frac{x + t - 1}{2(r - l)} \theta + 2\theta \right].$$

The integral over $(l, r) \in R \cap B^c$ of (7.3) is therefore bounded by

$$\begin{aligned} & \frac{C_\theta}{4\theta} e^{2\theta} \int_{l=(1-x+2t)/3}^t \int_{r=t}^{(x-1+3l)/2} \frac{1}{(r-l)^2} \exp \left[-\frac{x+t-1}{2(r-l)} \theta \right] dr dl \\ &= \frac{C_\theta}{4\theta} e^{2\theta} \int_{l=(1-x+2t)/3}^t \int_{s=t-l}^{(x-1+l)/2} \frac{1}{s^2} \exp \left(-\frac{x+t-1}{2} \theta s^{-1} \right) ds dl \\ &\leq \frac{C_\theta}{4\theta} e^{2\theta} \frac{2}{\theta(x+t-1)} \int_{l=(1-x+2t)/3}^t \exp \left(-\frac{x+t-1}{2} \theta \frac{2}{x-1+l} \right) dl \\ &\leq \frac{C_\theta}{2\theta^2} e^{2\theta} \frac{1}{x+t-1} e^{-\theta} \left(t - \frac{1-x+2t}{3} \right) = \frac{C_\theta}{6\theta^2} e^\theta < \infty. \end{aligned} \quad (7.6)$$

Term (ii). $R \cap B$. We can bound (7.3) by the sum of the integrals of the same

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integrand over the intervals I^* and

$$I' := \left\{ y : 0 < y \leq \frac{x+t-1}{2(r-l)} - 2 \right\}.$$

The bound for the integral over I^* is the same as the bound for the $R \cap B^c$ term. To bound the integral over I' , we first observe that

$$d_{l,r}^*(x, z, y) \leq \frac{1}{[\frac{1}{2}(x-t-1) + 2r-l]^2} \leq \frac{4}{(x+t-1)^2},$$

where the last inequality holds because $l < t < r$. The contribution to (7.2) can be bounded by integrating $4/(x+t-1)^2$ with respect to $(l, r) \in R \cap B$. We then extend this region of integration to R , and thus bound the contribution by

$$\begin{aligned} \frac{4}{(x+t-1)^2} \int_{l=\frac{2t+1-x}{3}}^t \left(\frac{x-1+3l}{2} - t \right) dl &\leq \frac{2}{(x+t-1)} \left(t - \frac{2t+1-x}{3} \right) \\ &= 2/3. \end{aligned}$$

This completes the proof for Subcase 1(a).

Subcase 1(b). $x - (1 + r - 2l) < y \leq z - (1 + r - 2l)$. First note that in this subcase we have $f^{(3)}(x-y) = 0$. We proceed in similar fashion as for Subcase 1(a), this time setting

$$I := \left\{ y : \frac{x-1+l}{r-l} - 2 < y \leq \frac{z-1+l}{r-l} - 2 \right\}.$$

Again using a linear change of variables, the integral (with respect to y only, in this subcase) appearing on the right in (7.1) in this subcase can be written as

$$\int_{y \in I} d_{l,r}^*(z, y) f_{\frac{t-l}{r-l}}(y) dy \tag{7.7}$$

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where now

$$d_{l,r}^*(z, y) = \frac{1}{z - (r - l)(y + 1) + 1 - r} \times \frac{2}{z - (r - l)(y + 1) + r - 1}.$$

Note that, unlike its analogue in Subcase 1(a), here $d_{l,r}^*(z, y)$ does not possess an explicit factor $z - x$.

By the same discussion as in Subcase 1(a), we are interested in the integral of (7.7) with respect to $(l, r) \in R$, where this time

$$R := \left\{ (l, r) : \frac{1 - z + 2t}{3} \vee 0 < l < t < r < 1 \wedge \frac{z - 1 + 3l}{2} \right\}.$$

and we may suppose that $z > 1 - t$.

Observe that on R we have

$$\frac{z - 1 + l}{r - l} - 2 > \frac{2}{3} \frac{z + t - 1}{r - l} - 2 > \frac{1}{2} \frac{z + t - 1}{r - l} - 2.$$

Following a line of attack similar to that for Subcase 1(a), we define

$$W := \left\{ (l, r) : \frac{x - 1 + l}{r - l} - 2 > \frac{z + t - 1}{2(r - l)} - 2 \right\}$$

and split our discussion of the integral of (7.7) over $(l, r) \in R$ into two terms, corresponding to (i) $R \cap W^c$ and (ii) $R \cap W$.

Term (i). $R \cap W$. We bound (7.7) by using the inequality (6.9) (for any $\theta > 0$) and

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obtain

$$\begin{aligned} & \int_{y \in I} \frac{1}{2-2l} \frac{1}{r-l} C_\theta \exp \left[-\theta \left(\frac{z+t-1}{2(r-l)} - 2 \right) \right] dy \\ & \leq \frac{1}{2} \frac{1}{1-t} \frac{1}{(r-l)^2} C_\theta e^{2\theta} \exp \left[-\theta \left(\frac{z+t-1}{2(r-l)} \right) \right] (z-x). \end{aligned}$$

Integrate this terms with respect to $(l, r) \in R \cap W$, we get no more than

$$(z-x) \frac{1}{2} \frac{1}{1-t} C_\theta e^{2\theta} \int_{l=(1-z+2t)/3}^t \int_{r=t}^{(z-1+3l)/2} \frac{1}{(r-l)^2} \exp \left[-\frac{z+t-1}{2(r-l)} \theta \right] dr dl,$$

which [consult (7.6)] is bounded by $(z-x)$ times a constant depending only on t and θ .

Term (ii). $R \cap W^c$. We partition the interval I of y -integration into the two subintervals

$$I^* := \left\{ y : \frac{z+t-1}{2(r-l)} - 2 < y \leq \frac{z-1+l}{r-l} - 2 \right\}$$

and

$$I' := \left\{ y : \frac{z-1+l}{r-l} - 2 < y \leq \frac{z+t-1}{2(r-l)} - 2 \right\}.$$

Observe that the length of each of the intervals I^* and I' is no more than the length of I , which is $(z-x)/(r-l)$. We can bound the integral over $y \in I^*$ and $(l, r) \in R \cap W^c$ just as we did for Term (i). For the integral over $y \in I'$ and $(l, r) \in R \cap W^c$, by plugging in the upper bound of y , we have the following inequality:

$$d_{l,r}^*(z, y) \leq \frac{1}{2-2t} \frac{2}{\frac{1}{2}(z+t-1) + 2r-l-t}.$$

Using the constant bound in Theorem 5.1, the integral of $d_{l,r}^*(z, y)$ with respect to

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$y \in I'$ and $(l, r) \in R \cap W^c$ is bounded above by

$$10 \frac{(z-x)}{1-t} \int_{l=(1-z+2t)/3}^t \int_{r=t}^{(z-1+3l)/2} \frac{1}{r-l} \frac{1}{\frac{1}{2}(z+t-1) + 2r-l-t} dr dl. \quad (7.8)$$

Write the integrand here in the form

$$\frac{1}{r-l} \frac{1}{\frac{z+t-1}{2} + 2r-l-t} = \left(\frac{1}{r-l} - \frac{2}{2r-l-t + \frac{z+t-1}{2}} \right) \frac{1}{l-t + \frac{z+t-1}{2}},$$

and observe that $l-t + \frac{z+t-1}{2} > \frac{z+t-1}{6} > 0$. Hence we can bound (7.8) by

$$\begin{aligned} & 10 \frac{(z-x)}{1-t} \int_{l=(1-z+2t)/3}^t \frac{1}{l-t + \frac{z+t-1}{2}} \left[\ln \frac{z-1+l}{2} - \ln(t-l) \right] dl \\ & \leq 10 \frac{(z-x)}{1-t} \frac{6}{z+t-1} \left[\frac{z+t-1}{3} \ln \frac{z+t-1}{2} - \int_{l=\frac{1-z+2t}{3}}^t \ln(t-l) dl \right] \\ & = 20 \frac{(z-x)}{1-t} \left(\ln \frac{z+t-1}{2} - \ln \left(\frac{z+t-1}{3} \right) + 1 \right) \\ & = 20 \left(1 + \ln \frac{3}{2} \right) \frac{(z-x)}{1-t}. \end{aligned}$$

This completes the proof for Subcase 1(b).

Subcase 1(c). $x - (1+r) < y \leq z - (1+r)$. In this case, the contribution from $f^{(3)}(z-y)$ vanishes. Without loss of generality we may suppose $z-x < t$, otherwise we can insert a factor $(z-x)/t$ in our upper bound, and the desired upper bound follows from the fact that the densities f_τ are all bounded by 10. Observe that the integrand $|d_{l,r}(x, z, y)|$ in the bound (7.1) is

$$\frac{1}{x-y+1-r} \frac{2}{x-y+r-1} \leq \frac{1}{x-z+2} \frac{2}{x-z+2r} \leq \frac{1}{2-t} \frac{2}{2r-t} \leq \frac{2}{t(2-t)}.$$

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Integrate this constant bound directly with respect to

$$P(Y \in dy | (L_3, R_3) = (l, r)) dr dl$$

on the region $x - (1 + r) < y \leq z - (1 + r)$ and $0 < l < t < r < 1$ and use the fact that the density is bounded by 10; we conclude that this contribution is bounded by $(z - x)$ times a constant that depends on t . This completes the proof for Subcase 1(c) and also for Case 1.

Case 2. $z - x \geq 2l$. In this case we simply use

$$|d_{l,r}(x, z, y)| \leq f^{(3)}(z - y) + f^{(3)}(x - y),$$

and show that each of the two terms on the right contributes at most a constant (depending on t) times $(z - x)$ to the bound in (7.1). Accordingly, let w be either x or z . We are interested in bounding

$$\int_{l=0}^{\frac{z-x}{2} \wedge t} \int_{r=t}^1 \int_{y=w-(1+r)}^{w-(1+r-2l)} \frac{1}{w-y+1-r} \frac{2}{w-y+r-1} \mu(dy, dr, dl) \quad (7.9)$$

with $\mu(dy, dr, dl) := P(Y \in dy | (L_3, R_3) = (l, r)) dr dl$. We bound the integrand as follows:

$$\frac{1}{w-y+1-r} \frac{2}{w-y+r-1} \leq \frac{1}{2-2l} \frac{2}{2r-2l} \leq \frac{1}{2} \frac{1}{1-t} \frac{1}{r-l}.$$

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We first suppose $z - x < t$ and bound (7.9) by

$$\begin{aligned} \frac{1}{2} \frac{1}{1-t} \int_{l=0}^{\frac{z-x}{2}} \int_{r=t}^1 \frac{1}{r-l} dr dl &\leq \frac{1}{2} \frac{1}{1-t} \int_{l=0}^{\frac{z-x}{2}} [-\ln(t-l)] dl \\ &\leq \frac{1}{2} \frac{1}{1-t} \left[-\ln \left(t - \frac{z-x}{2} \right) \right] \frac{z-x}{2} \\ &\leq (z-x) \frac{\ln(2/t)}{4(1-t)}. \end{aligned}$$

If instead $z - x \geq t$, we bound (7.9) by

$$\frac{1}{2} \frac{z-x}{t} \int_{l=0}^t \int_{r=t}^1 \frac{1}{1-l} \frac{1}{r-l} dr dl \leq (z-x) \frac{\pi^2}{12t}.$$

This completes the proof for Case 2 and thus the proof of Lipschitz continuity of $c^{(3)}$. □

We immediately get the following corollary from the proof of Theorem 7.1.

Corollary 7.4. *For any $0 < \eta < 1/2$, the uniform continuous family $\{f_t : t \in [\eta, 1 - \eta]\}$ is a uniformly equicontinuous family.*

Proof. We observe from the proof of Theorem 7.1 that for any $0 < \eta < 1/2$, the Lipschitz constants L_t in Theorem 7.1 are bounded for $t \in [\eta, 1 - \eta]$ by some universal constant $C < \infty$. The result follows. □

Remark 7.5. The uniform equicontinuity in Corollary 7.4 does not hold for the family $\{f_t : t \in (0, 1)\}$. Here is a proof. For the sake of contradiction, suppose to the contrary. We symmetrize $f_t(x)$ at $x = 0$ for every $0 \leq t \leq 1$ to create another family of continuous densities g_t ; that is, consider $g_t(x) := [f_t(x) + f_t(-x)]/2$. Observe that the supposed uniform equicontinuity of the functions f_t for $t \in (0, 1)$ extends to the functions g_t . Now suppose (for each $t \in [0, 1]$) that $W(t)$ is a random variable

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with density g_t . By a simple calculation we have $W(t) \Rightarrow W(0)$, and it follows by Boos [4, Lemma 1] that $g_t(x) \rightarrow g_0(x)$ uniformly in x . This contradicts to the fact that $g_t(0) = 0$ for all $t \in (0, 1)$ but $g_0(0) = e^{-\gamma}$.

Remark 7.6. Since $(F_t)_{t \in [0,1]}$ is weakly continuous in t and F_t is atomless for $0 \leq t \leq 1$, it follows from a theorem of Pólya ([5, Exercise 4.3.4]) that $(F_t)_{t \in [0,1]}$ is continuous in the sup-norm metric, i.e., that $(J(t))$ [or $(Z(t))$] is continuous in the Kolmogorov–Smirnov metric on distributions.

7.2 Joint continuity of $f_t(x)$ in (t, x)

In this section, we prove that the continuous density functions $f_t(x)$ are jointly continuous for $(t, x) \in (0, 1) \times \mathbb{R}$. As noted in the proof of Lemma 6.2, we reference Grübel and Rösler [21] to conclude that for each $t \in [0, 1)$, the distribution functions F_u converge weakly to F_t as $u \downarrow t$. It follows by symmetry that the convergence also holds for each $t \in (0, 1]$ as $u \uparrow t$. We now deduce the convergence from f_u to f_t for each $t \in (0, 1)$ as $u \rightarrow t$, according to the following lemma.

Lemma 7.7. *For each $0 < t < 1$ we have $f_u \rightarrow f_t$ uniformly as $u \rightarrow t$.*

Proof. We fix $0 < t \leq 1/2$ and choose $0 < \eta < t$. By the weak convergence of F_u to F_t as $u \rightarrow t$, the uniform boundedness of the density functions (Theorem 5.1), the fact that $f_t(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, and the uniform equicontinuity of the family $\{f_u : u \in [\eta, 1 - \eta]\}$ (Corollary 7.4), we conclude from Boos [4, Lemma 1] (a converse to Scheffé’s theorem) that $f_u \rightarrow f_t$ uniformly as $u \rightarrow t$. \square

Corollary 7.8. *The density $f_t(x)$ is jointly continuous in $(t, x) \in (0, 1) \times \mathbb{R}$.*

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Proof. As $(t', x') \rightarrow (t, x) \in (0, 1) \times \mathbb{R}$, we have

$$\begin{aligned} \limsup |f_{t'}(x') - f_t(x)| &\leq \limsup |f_{t'}(x') - f_t(x')| + \limsup |f_t(x') - f_t(x)| \\ &\leq \limsup \|f_{t'} - f_t\|_\infty + \delta_t(|x' - x|) \\ &= 0 \end{aligned}$$

where the sup-norm $\|f_{t'} - f_t\|_\infty$ tends to 0 as $t' \rightarrow t$ by Lemma 7.7 and the modulus of uniform continuity δ_t of the function f_t tends to 0 as $x' \rightarrow x$ by Theorem 5.11. \square

Remark 7.9. The positivity of $f_t(x)$ for each $0 < t < 1$ and $x > \min\{t, 1-t\}$ in Theorem 6.9 can be proved alternatively by using the integral equation Proposition 6.7 and the joint continuity result of Corollary 7.8. Here is the proof.

Fix (for now) $t_0, t_1, t_2 \in (0, 1)$ with $t_1 > t_0 > t_2$. We will show that $f_{t_0}(x) > 0$ for all $x > t_0$, using t_1 and t_2 in auxiliary fashion. Since this is true for arbitrarily chosen t_0 , invoking symmetry ($f_t \equiv f_{1-t}$) then completes the proof.

We certainly know that $f_{t_0}(y_0) > 0$ for some $y_0 > t_0$; choose and fix such a y_0 . Use Proposition 6.7 to represent the density $f_{t_1}(x)$. We observe that the integrand of the integral with respect to l is positive at $l = l_1 = (t_1 - t_0)/(1 - t_0)$ and $x = y_1 = (1 - l_1)(y_0 + 1)$. From Corollary 7.8 we conclude that the integrand is positive in a neighborhood of l_1 and thus $f_{t_1}(y_1) > 0$.

Further, use Proposition 6.7 to represent the density $f_{t_0}(x)$. We observe that the integrand of the integral with respect to r is positive at $r = r_2 = \frac{t_0}{t_1}$ and $x = y_2 = r_2(y_1 + 1)$. From $f_{t_1}(y_1) > 0$ and Corollary 7.8 we conclude that $f_{t_0}(y_2) > 0$.

Now letting $y_2 = y_0 + \epsilon_1$, we have

$$\epsilon_1 = \left(\frac{t_0}{t_1} \frac{1 - t_1}{1 - t_0} - 1 \right) y_0 + \frac{t_0}{t_1} \left(1 + \frac{1 - t_1}{1 - t_0} \right).$$

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Observe that as $t_1 \downarrow t_0$ we have $\epsilon_1 \rightarrow 2$, while as $t_1 \uparrow 1$ we have $\epsilon_1 \downarrow -y_0 + t_0 < 0$. Thus, given $\delta \in (0, 2 - t_0 + y_0)$ it is possible to choose $t_1 \in (t_0, 1)$ such that $\epsilon_1 = -y_0 + t_0 + \delta$, i.e., $y_2 = t_0 + \delta$. We conclude that $f_{t_0}(x)$ is positive for every $x > t_0$, as desired.

Chapter 8

Right tail asymptotics for F_t and f_t ; large deviations for QuickQuant

8.1 Improved right-tail asymptotic upper bounds for F_t and f_t

In this section, we will prove that for $0 < t < 1$ and $x > 4$, the continuous density function f_t satisfies

$$f_t(x) \leq \exp[-x \ln x - x \ln \ln x + O(x)]$$

uniformly in t . We first bound the moment generating function of the random variable V treated in Lemma 5.3.

Lemma 8.1. *Denote the moment generating function of V by m . Then for every*

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$\epsilon > 0$ there exists a constant $a \equiv a(\epsilon) > 0$ such that for all $\theta > 0$ we have

$$m(\theta) \leq \exp[(2 + \epsilon)\theta^{-1}e^\theta + a\theta]. \quad (8.1)$$

Proof. The idea of the proof comes from Janson [24, Lemma 6.1]. Observe that the random variable V satisfies the following distributional identity

$$V \stackrel{\mathcal{L}}{=} 1 + V_1 \cdot V$$

where $V_1 \sim \text{Uniform}(1/2, 1)$ is independent of V . It follows by conditioning on V_1 that the moment generating function m satisfies

$$m(\theta) = 2e^\theta \int_{v=1/2}^1 m(\theta v) dv = 2e^\theta \int_{u=0}^{1/2} m(\theta(1-u)) du. \quad (8.2)$$

Since m is continuous and $m(0) = 1$, there exists a $\theta_1 > 0$ such that the inequality (8.1) holds (for *any* constant $a > 0$) for $\theta \in [0, \theta_1]$. Choose and fix $\theta_2 > \max\{\theta_1, 5\}$ and choose $a \in [1, \infty)$ large enough such that the inequality (8.1) holds for $\theta \in [\theta_1, \theta_2]$.

We now suppose for the sake of contradiction that (8.1) fails at some $\theta > \theta_2$. Define $T := \inf\{\theta > \theta_2 : (8.1) \text{ fails}\}$; then by continuity we have $m(T) = \exp[(2 + \epsilon)T^{-1}e^T + aT]$. Since $m(\theta u) \geq 1$ for any $\theta > 0$ and $0 < u < 1/2$, we can conclude from (8.2) that m satisfies

$$m(\theta) \leq 2e^\theta \int_{u=0}^{1/2} m(\theta u) m(\theta(1-u)) du \quad (8.3)$$

for every $\theta > 0$, including for $\theta = T$. The proof is now completed effortlessly by applying exactly the same argument as for the limiting **QuickSort** moment generating function in Lemma 3.4; indeed, using only (8.3) we prove there that when

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$\theta = T$ the right-hand side of (8.3) is strictly smaller than $m(T)$, which is the desired contradiction. \square

Thus, for $\epsilon > 0$ and $\theta > 0$, the moment generating functions m_t all satisfy

$$m_t(\theta) \leq m(\theta) \leq \exp[(2 + \epsilon)\theta^{-1}e^\theta + a\theta]. \quad (8.4)$$

We now deduce a uniform right-tail upper bound on the survival functions $1 - F_t$ for $0 < t < 1$.

Theorem 8.2. *Uniformly in $0 < t < 1$, for $x > 1$ the distribution function F_t for $J(t)$ satisfies*

$$1 - F_t(x) \leq \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. The proof is essentially the same as for Proposition 3.2, but for completeness we sketch the simple proof here. Fix $\epsilon > 0$. For any $\theta > 0$ we have the Chernoff bound

$$\begin{aligned} 1 - F_t(x) &= \mathbb{P}(J(t) > x) \leq \mathbb{P}(Z(t) > x) \\ &\leq e^{-\theta x} m_t(\theta) \leq e^{-\theta x} \exp[(2 + \epsilon)\theta^{-1}e^\theta + a\theta] \end{aligned}$$

by (8.4). Letting $\theta = \ln[(2 + \epsilon)^{-1}x \ln x]$, and then $\epsilon \downarrow 0$ we get the desired upper bound—in fact, with the following improvement we will not find useful in the sequel:

$$1 - F_t(x) \leq \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)x + o(x)].$$

\square

The continuous density function $f_t(x)$ enjoys the same uniform asymptotic bound

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for $0 < t < 1$ and $x > 4$.

Theorem 8.3. *Uniformly in $0 < t < 1$, for $x > 4$ the continuous density function f_t satisfies*

$$f_t(x) \leq \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. Fix $0 < t < 1$ and let $x > 4$. We first use the integral equation in Proposition 6.8, namely,

$$f_t(x) = \int \mathbb{P}((L_3(t), R_3(t)) \in d(l, r)) \cdot h_t(x \mid l, r),$$

for $x \geq 0$, where, by a change of variables,

$$h_t(x \mid l, r) = \int f_{l,r}((r-l)(y-1)) f_{\frac{t-l}{r-l}}\left(\frac{x}{r-l} - y\right) dy; \quad (8.5)$$

we consider the contribution to $f_t(x)$ from values (l, r) satisfying $0 < l < t < r < 1$. Recall that the conditional density $f_{l,r}(z)$ vanishes if $z \geq 2$. Thus the only nonzero contribution to (8.5) is from values of y satisfying

$$y \leq \frac{2}{r-l} + 1.$$

If this inequality holds, then the argument for the factor $f_{(t-l)/(r-l)}$ satisfies

$$\frac{x}{r-l} - y \geq \frac{x-2}{r-l} - 1 \geq x-3.$$

Using $b(l, r)$ of Lemma 5.2 and (5.3) to bound the $f_{l,r}$ factor, we obtain

$$h_t(x \mid l, r) \leq b(l, r) \left(1 - F_{\frac{t-l}{r-l}}(x-3)\right).$$

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By Theorem 8.2 and the last display in the proof of Lemma 5.2, the contribution in question is thus bounded by $\exp[-x \ln x - x \ln \ln x + O(x)]$, uniformly in t , for $x > 4$.

For the contribution to $f_t(x)$ corresponding to the cases $0 = L_3(t) < t < R_3(t) < 1$ and $0 < L_3(t) < t < R_3(t) = 1$, we use the same idea as in the proof of Lemma 5.6. By symmetry, we need only consider the first of these two cases. Recall from the proof of Lemma 5.6 that the contribution in question is bounded by the sum of $f_W(x)$, which is the density of $W = U_1(1 + U_2V)$ evaluated at x [where U_1 , U_2 , and V are independent, U_1 and U_2 are uniformly distributed on $(0, 1)$, and V is as in Lemma 5.3], and the integral

$$\begin{aligned} \int_{r=0}^1 r^{-1} \mathbb{P}\left(V > \frac{x}{r} - 1\right) dr &= \int_{v=x-1}^{\infty} (v+1)^{-1} \mathbb{P}(V > v) dv \\ &\leq x^{-1} \int_{v=x-1}^{\infty} \mathbb{P}(V > v) dv \\ &\leq \exp[-x \ln x - x \ln \ln x + O(x)]. \end{aligned}$$

The last inequality here is obtained by applying a Chernoff bound and Lemma 8.1 to the integrand and integrating; we omit the straightforward details. To bound the density of W at x , observe that by conditioning on the values of U_2 and V , we have

$$\begin{aligned} f_W(x) &= \int_{u,v} (1 + uv)^{-1} \mathbb{1}(0 \leq x \leq 1 + uv) \mathbb{P}(U_2 \in du, V \in dv) \\ &= \int_{u=0}^1 \int_{v=(x-1)/u}^{\infty} (1 + uv)^{-1} \mathbb{P}(V \in dv) du \\ &\leq x^{-1} \int_{u=0}^1 \mathbb{P}\left(V > \frac{x-1}{u}\right) du \\ &\leq x^{-1} \mathbb{P}(V > x-1) \leq \exp[-x \ln x - x \ln \ln x + O(x)]. \end{aligned}$$

This completes the proof. □

8.2 Matching right-tail asymptotic lower bounds for F_t and f_t

In this section we will prove for each fixed $t \in (0, 1)$ that the continuous density function f_t satisfies

$$f_t(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)] \text{ as } x \rightarrow \infty,$$

matching the upper bound of Theorem 8.2 to two logarithmic asymptotic terms, with remainder of the same order of magnitude. While we are able to get a similarly matching lower bound to Theorem 8.3 for the survival function $1 - F_t$ that is uniform in t , we are unable to prove uniformity in t for the density lower bound.

We begin with consideration of the survival function.

Theorem 8.4. *Uniformly in $0 < t < 1$, the distribution function F_t for $J(t)$ satisfies*

$$1 - F_t(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. With D denoting a random variable having the Dickman distribution with support $[1, \infty)$, for any $0 < t < 1$ we have from Lemma 5.4 that

$$\begin{aligned} 1 - F_t(x) &= \mathbb{P}(J(t) > x) = \mathbb{P}(Z(t) > x + 1) \geq \mathbb{P}(D > x + 1) \\ &= \exp[-x \ln x - x \ln \ln x + O(x)] \text{ as } x \rightarrow \infty. \end{aligned}$$

The asymptotic lower bound here follows by substitution of $x+1$ for u in equation (1.6) (for the unnormalized Dickman function) of Xuan [37], who credits earlier work of de Bruijn [6] and of Hua [22]. □

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Now we turn our attention to the densities.

Theorem 8.5. *For each fixed $t \in (0, 1)$ we have*

$$f_t(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)] \text{ as } x \rightarrow \infty.$$

Proof. From the calculations at the beginning of the proof of Lemma 6.14, for all $z > 0$ we have

$$f_t(t + z) \geq z \iint_{\substack{v > u > 0: u+v < 1, \\ u < (1-t)/z, v-u < t/z}} (t + uz)^{-1} v^{-1} f_{1-\frac{u}{v}} \left(\frac{1-u-v}{v} \right) du dv.$$

Thus, changing variables from u to $w = 1 - (u/v)$, we have

$$f_t(t + tz) \geq z \int_0^1 \int_0^{\Upsilon(t,z,w)} [1 + v(1-w)z]^{-1} f_w(v^{-1} + w - 2) dv dw,$$

where $\Upsilon(t, z, w) := \min\{(2-w)^{-1}, (1-t)(tz)^{-1}(1-w)^{-1}, z^{-1}w^{-1}\}$. Now let

$$\Lambda(t, z, w) := \max\{0, t(1-t)^{-1}z(1-w) + w - 2, zw + w - 2\}$$

and change variables from v to $s = v^{-1} + w - 2$ to find

$$\begin{aligned} f_t(t + tz) &\geq z \int_0^1 \int_{\Lambda(t,z,w)}^\infty [1 + (2-w+s)^{-1}(1-w)z]^{-1} (2-w+s)^{-2} f_w(s) ds dw. \end{aligned}$$

Observe that if $\delta > 0$ and $t \leq w \leq (1 + \delta)t \leq 1$, then

$$\Lambda(t, z, w) < (1 + \delta)tz$$

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and so

$$\begin{aligned} f_t(t + tz) &\geq z \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{\infty} [1 + (2 - w + s)^{-1}(1 - w)z]^{-1} (2 - w + s)^{-2} f_w(s) \, ds \, dw. \end{aligned}$$

If $\delta \leq 1$, it follows that

$$\begin{aligned} f_t(t + tz) &\geq z \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} [1 + (2 - w + s)^{-1}(1 - w)z]^{-1} (2 - w + s)^{-2} f_w(s) \, ds \, dw \\ &\geq \frac{z}{(1 + 2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} \frac{1}{1 + (2 - w + (1 + \delta)tz)^{-1}(1 - w)z} f_w(s) \, ds \, dw \\ &\geq \frac{z}{(1 + 2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} \left[1 + \frac{1 - w}{(1 + \delta)t} \right]^{-1} f_w(s) \, ds \, dw \\ &\geq \frac{z}{(1 + 2tz)^2} \frac{(1 + \delta)t}{1 + \delta t} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} f_w(s) \, ds \, dw \\ &\geq \frac{tz}{(1 + 2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} f_w(s) \, ds \, dw \\ &= \frac{tz}{(1 + 2tz)^2} \int_t^{(1+\delta)t} [\mathbb{P}(J(w) > (1 + \delta)tz) - \mathbb{P}(J(w) > 2tz)] \, dw \end{aligned}$$

Recall that D defined in Lemma 5.4 is a random variable having the Dickman distribution and V defined in (5.5). By Lemma 5.3 and Lemma 5.4, we have the relationship that $D - 1 \leq J(w) \leq V - 1$ stochastically, and thus we can further lower bound the density function as

$$\begin{aligned} f_t(t + tz) &\geq \frac{tz}{(1 + 2tz)^2} \int_t^{(1+\delta)t} [\mathbb{P}(D - 1 > (1 + \delta)tz) - \mathbb{P}(V > 2tz)] \, dw \\ &= \delta t \frac{tz}{(1 + 2tz)^2} [\mathbb{P}(D - 1 > (1 + \delta)tz) - \mathbb{P}(V > 2tz)]. \end{aligned}$$

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That is, if $0 < \delta \leq \min\{1, t^{-1} - 1\}$, then for every $z > 0$ we have

$$f_t(t + z) \geq \delta t \frac{z}{(1 + 2z)^2} [\mathbb{P}(D - 1 > (1 + \delta)z) - \mathbb{P}(V > 2z)].$$

If $z \geq \max\{1, t/(1 - t)\}$, then we can choose $\delta \equiv \delta_z = z^{-1}$ and conclude

$$f_t(t + z) \geq t(1 + 2z)^{-2} [\mathbb{P}(D - 1 > z + 1) - \mathbb{P}(V > 2z)].$$

Moreover, as $z \rightarrow \infty$, we have

$$(1 + 2z)^{-2} [\mathbb{P}(D - 1 > z + 1) - \mathbb{P}(V > 2z)] = \exp[-z \ln z - z \ln \ln z + O(z)].$$

The stated result follows readily. □

Remark 8.6. The proof of Theorem 8.4 reveals that the result in fact holds uniformly for t in any closed subinterval of $(0, 1)$. In fact, the proof shows that the result follows uniformly in $t \in (0, 1)$ and $x \rightarrow \infty$ satisfying $x = \Omega(\ln[1/\min\{t, 1 - t\}])$.

8.3 Right-tail large deviations for QuickQuant

In this section, we investigate the right-tail large deviation behavior of $\text{QuickQuant}(n, t)$, that is, of $\text{QuickSelect}(n, m_n(t))$. Throughout this section, for each fixed $0 \leq t \leq 1$ we consider any sequence $1 \leq m_n(t) \leq n$ such that $m_n(t)/n \rightarrow t$ as $n \rightarrow \infty$. We abbreviate the normalized number of key comparisons of $\text{QuickSelect}(n, m_n(t))$ discussed in Section 4.1 as $C_n(t) := n^{-1}C_{n, m_n(t)}$.

Kodaj and Móri [27, Corollary 3.1] bound the convergence rate of $C_n(t)$ to its limit $Z(t)$ in the Wasserstein d_1 -metric, showing that the distance is $O(\delta_{n,t} \log(\delta_{n,t}^{-1}))$,

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where $\delta_{n,t} = |n^{-1}m_n(t) - t| + n^{-1}$. Using their result, we bound the convergence rate in Kolmogorov–Smirnov distance in the following lemma .

Lemma 8.7. *Let $d_{\text{KS}}(\cdot, \cdot)$ be Kolmogorov–Smirnov (KS) distance. Then*

$$d_{\text{KS}}(C_n(t), Z(t)) = \exp \left[-\frac{1}{2} \ln \frac{1}{\delta_{n,t}} + \frac{1}{2} \ln \ln \frac{1}{\delta_{n,t}} + O(1) \right]. \quad (8.6)$$

Proof. The lemma is an immediate consequence of Fill and Janson [16, Lemma 5.1], since the random variable $Z(t)$ has a density function bounded by 10, according to Theorem 5.1. Indeed, by that result we have

$$d_{\text{KS}}(C_n(t), Z(t)) \leq 2^{1/2} [10 d_1(C_n(t), Z(t))]^{1/2} = O([\delta_{n,t} \log(\delta_{n,t}^{-1})]^{1/2}).$$

□

Using the right-tail asymptotic bounds on the limiting **QuickQuant**(t) distribution function F_t in Theorems 8.2 and 8.4 (which extend to $t \in \{0, 1\}$ by known results about the Dickman distribution), we can now derive the right-tail large-deviation behavior of $C_n(t)$.

Theorem 8.8. *Fix $t \in [0, 1]$ and abbreviate $\delta_{n,t}$ as δ_n . Let (ω_n) be any sequence diverging to $+\infty$ as $n \rightarrow \infty$ and let $c > 1$. For integer $n \geq 3$, consider the interval*

$$I_n := \left[c, \frac{1}{2} \frac{\ln \delta_n^{-1}}{\ln \ln \delta_n^{-1}} \left(1 - \frac{\omega_n}{\ln \ln \delta_n^{-1}} \right) \right].$$

(a) *Uniformly for $x \in I_n$ we have*

$$\mathbb{P}(C_n(t) > x) = (1 + o(1)) \mathbb{P}(Z(t) > x) \quad \text{as } n \rightarrow \infty. \quad (8.7)$$

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(b) If $x_n \in I_n$ for all large n , then

$$\mathbb{P}(C_n(t) > x_n) = \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)]. \quad (8.8)$$

Proof. The proof is similar to Fill and Hung [13, Theorem 3.3] or its improvement in Theorem 3.16. We prove part (a) first. By Lemma 8.7, it suffices to show that

$$\exp \left[-\frac{1}{2} \ln \frac{1}{\delta_n} + \frac{1}{2} \ln \ln \frac{1}{\delta_n} + O(1) \right] \leq o(\mathbb{P}(Z(t) > x_n))$$

with $x_n \equiv \frac{1}{2} \frac{\ln \delta_n^{-1}}{\ln \ln \delta_n^{-1}} \left(1 - \frac{\omega_n}{\ln \ln \delta_n^{-1}} \right)$ and $\omega_n = o(\ln \ln \delta_n^{-1})$. Since, by Theorem 8.4, we have

$$\mathbb{P}(Z(t) > x_n) \geq \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)],$$

it suffice to show that for any constant $C < \infty$ we have

$$-\frac{1}{2} \ln \frac{1}{\delta_n} + \frac{1}{2} \ln \ln \frac{1}{\delta_n} + C + x_n \ln x_n + x_n \ln \ln x_n + Cx_n \rightarrow -\infty. \quad (8.9)$$

This is routine and similar to what is done in [13, proof of Theorem 3.3]. Writing L for \ln and L_k for the k th iterate of L , and abbreviating $\alpha_n := 1 - \frac{\omega_n}{L_2 \delta_n^{-1}}$, we have

$$x_n(L x_n + L_2 x_n + C) = \frac{1}{2} \frac{L \frac{1}{\delta_n}}{L_2 \frac{1}{\delta_n}} \beta_n,$$

where

$$\beta_n = \alpha_n \left[\left(L_2 \frac{1}{\delta_n} - L_3 \frac{1}{\delta_n} - L_2 + L \alpha_n \right) + L \left(L_2 \frac{1}{\delta_n} - L_3 \frac{1}{\delta_n} - L_2 + L \alpha_n \right) + C \right].$$

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For n large,

$$\begin{aligned}
 \beta_n &= \alpha_n \left[L_2 \frac{1}{\delta_n} + C - L_2 + L \alpha_n + L \left(1 - \frac{L_3 \frac{1}{\delta_n} + L_2 - L \alpha_n}{L_2 \frac{1}{\delta_n}} \right) \right] \\
 &= \alpha_n \left[L_2 \frac{1}{\delta_n} + C - L_2 + L \alpha_n - (1 + o(1)) \frac{L_3 \frac{1}{\delta_n}}{L_2 \frac{1}{\delta_n}} \right] \\
 &= \alpha_n \left[L_2 \frac{1}{\delta_n} + C - L_2 + o(1) \right],
 \end{aligned}$$

and therefore

$$\begin{aligned}
 &x_n(L x_n + L_2 x_n + C) \\
 &= \left(\frac{1}{2} L \frac{1}{\delta_n} \right) \alpha_n \left[1 + \frac{C - L_2 + o(1)}{L_2 \frac{1}{\delta_n}} \right] = \frac{1}{2} L \frac{1}{\delta_n} - (1 + o(1)) \omega_n \frac{L \frac{1}{\delta_n}}{2 L_2 \frac{1}{\delta_n}}.
 \end{aligned}$$

Thus the expression in (8.9) equals

$$-(1 + o(1)) \omega_n \frac{L \frac{1}{\delta_n}}{2 L_2 \frac{1}{\delta_n}} + \frac{1}{2} L_2 \frac{1}{\delta_n} + C = -(1 + o(1)) \omega_n \frac{L \frac{1}{\delta_n}}{2 L_2 \frac{1}{\delta_n}} \rightarrow -\infty.$$

This completes the proof of part (a).

Part (b) is immediate from part (a) and Theorems 8.2 and 8.4. □

Remark 8.9. Consider the particular choice $m_n(t) = \lfloor nt \rfloor + 1$ of the sequences $(m_n(t))$ for $t \in [0, 1)$, with $m_n(1) = n$. That is, suppose that $C_n(t) = X_n(t)$ as defined in (4.6). In this case, large-deviation upper bounds based on tail estimates of the limiting F_t have broader applicability than as described in Theorem 8.8 and are easier to derive, too. The reason is that, by Kodaj and Móri [27, Lemma 2.4], the random variable $X_n(t)$ is stochastically dominated by its continuous counterpart

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$Z(t)$. Then, by Theorem 8.4, uniformly in $t \in [0, 1]$, we have

$$\mathbb{P}(X_n(t) > x) \leq \mathbb{P}(Z(t) > x) \leq \exp[-x \ln x - x \ln \ln x + O(x)] \quad (8.10)$$

for $x > 1$; there is *no restriction at all* on how large x can be in terms of n or t .

Here is an example of a *very* large value of x for which the tail probability is nonzero and the aforementioned bound still matches logarithmic asymptotics to lead order of magnitude, albeit not to lead-order term. The largest possible value for the number $C_{n,m}$ of comparisons needed by **QuickSelect**(n, m) is $\binom{n}{2}$, corresponding in the natural coupling to any permutation of the n keys for which the $m-1$ keys smaller than the target key appear in increasing order, the $n-m$ keys larger than the target key appear in decreasing order, and the target key appears last; thus

$$\mathbb{P}\left(C_{n,m} = \binom{n}{2}\right) = \frac{1}{n!} \times \binom{n-1}{m-1},$$

which lies between $1/n!$ and $\binom{n-1}{\lceil (n-1)/2 \rceil} / n! \sim 2^{n-(1/2)} / (n! \sqrt{\pi n})$. We conclude that for $x_n = (n-1)/2$ we have, uniformly in $t \in [0, 1]$, that

$$P(X_n(t) \geq x_n) = \mathbb{P}(X_n(t) = x_n) = \exp[-2x_n \ln x_n + O(x_n)].$$

The bound (8.10) on $\mathbb{P}(X_n(t) > x)$ is in fact also (by the same proof) a bound on the larger probability $\mathbb{P}(X_n(t) \geq x)$, and in this case implies

$$P(X_n(t) \geq x_n) = \exp[-x_n \ln x_n + O(x_n \log \log x_n)].$$

The bound (8.10) is thus loose only by an asymptotic factor of 2 in the logarithm of the tail probability.

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Remark 8.10. (a) We can use another result of Kodaj and Móri, namely, [27, Lemma 3.2], in similar fashion to quantify the Kolmogorov–Smirnov continuity of the process Z discussed in Remark 7.6. Let $0 \leq t < u \leq 1/2$ and $\delta = u - t$. Then the lemma asserts

$$d_1(Z(t), Z(u)) < 4\delta(1 + 2\log \delta^{-1}).$$

It follows using Fill and Janson [16, Lemma 5.1] that

$$d_{\text{KS}}(Z(t), Z(u)) \leq O((\delta \log \delta^{-1})^{1/2}) = \exp \left[-\frac{1}{2} \ln \delta^{-1} + \frac{1}{2} \ln \ln \delta^{-1} + O(1) \right],$$

uniformly for $|u - t| \leq \delta$, as $\delta \downarrow 0$. We thus have *uniform* Kolmogorov–Smirnov continuity of Z .

(b) Kodaj and Móri [27] did not consider a lower bound on either of the distances in (a), but we can rather easily obtain a lower bound on the KS distance that is of order δ^2 uniformly for t and u satisfying $0 < t < t + \delta = u \leq \min\{1/2, 2t\}$.

Indeed, for such t and u we have $\mathbb{P}(J(u) \leq u) = 0$ and, by Theorem 6.15 (since $t \leq u \leq 1/2 \leq \min\{1 - t, 2t\}$, as required by the hypotheses of the theorem) and in the notation of that theorem,

$$\begin{aligned} \mathbb{P}(J(t) \leq u) &= \int_t^u f_t(x) dx = t \int_0^{(u/t)-1} \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k dz \\ &\geq t \int_0^{(u/t)-1} (c_1 z - c_2 z^2) dz \\ &= t \left[\frac{1}{2} c_1 \left(\frac{u}{t} - 1 \right)^2 - \frac{1}{3} c_2 \left(\frac{u}{t} - 1 \right)^3 \right] \\ &\geq \frac{1}{3} c_1 t \left(\frac{u}{t} - 1 \right)^2 > \frac{1}{150} (u - t)^2 = \frac{1}{150} \delta^2, \end{aligned}$$

where the penultimate inequality holds because $\frac{u}{t} - 1 = \frac{\delta}{t} < 1$ and $0 < c_2 \leq \frac{1}{2} c_1$.

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(c) The lower bound in (b) can be improved to order δ when $t = 0$. Then for every $u \in [0, 1]$ we have $\mathbb{P}(J(0) \leq u) = e^{-\gamma u}$, and so for $u \in [0, 1/2]$ we have

$$d_{\text{KS}}(Z(0), Z(u)) \geq e^{-\gamma u}.$$

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Vita

Wei-Chun Hung

Contact Information

Applied Mathematics and Statistics

Johns Hopkins University

3400 North Charles Street

E-mail: whung6@jhu.edu

Baltimore, MD 21218

Research Interests

Analysis of algorithms, probability theory, stochastic processes

Education

Johns Hopkins University, Baltimore, MD USA

- PhD., Applied Mathematics and Statistics, August 2021
- M.S., Applied Mathematics and Statistics, May 2021

National Taiwan University, Taipei, Taiwan

- M.S., Financial Engineering, June 2012

VITA

- B.A., Mathematics, June 2010

Publications

- James Allen Fill, **Wei-Chun Hung** “*QuickSort: Improved right-tail asymptotics for the limiting distribution, and large deviations*” Electronic Journal of Probability, 24:Paper No. 67, 13 pages, 2019
- James Allen Fill, **Wei-Chun Hung** “*QuickSort: Improved right-tail asymptotics for the limiting distribution, and large deviations (Extended Abstract)*” In 2019 Proceedings of the Sixteenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO), pages 87–93. SIAM, Philadelphia, PA, 2019
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Awards

Johns Hopkins University

- **Charles and Catherine Counselman Fellowship**, 2018–2021
- **Professor Joel Dean Awards for Excellence in Teaching**, 2017, 2018, and 2019

VITA

- **AMS Research Fellowship**, 2019
- **Whiting School of Engineering's Teaching Fellow Recognition Award**, 2017
- **Applied Mathematics and Statistics Department Fellowship**, 2015–2018

National Taiwan University

- **Presidential Award**, Spring 2010, Fall 2010

Teaching Experience

Johns Hopkins University, Baltimore, MD USA

- Teaching Assistant September 2015 to May 2021
 - Instructed students in the areas of Statistics, Probability Theory, Machine Learning, Optimization, Mathematical Biology, Game Theory and Discrete Math